Recent Applications of Generalized Mathematical Analysis

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Abstract: There are presented two applications of pseudo-analysis in the theory of random sets and nonlinear partial differential equations.

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1 Introduction

In the paper [39] there were presented many different applications of generalized mathematical analysis, so called pseudo-analysis, see [33, 36]. In this paper we present two recent applications of the pseudo-analysis.

First application is related to random sets. Kendall ([17]) and Matheron ([23]) laid down the theoretical foundations of the theory of random sets. This theory is based on probability measures on the space of closed subsets of locally compact Hausdorff and separable space endowed with hit-or-miss topology. Random closed sets has been introduced as generalizations of random variables, i.e., random closed sets are random elements on the space of closed subsets. The mathematical foundation of random closed sets is based on Choquet's capacity theorem ([8]) and the special properties of random closed sets follow from the topological structure of the space of closed sets. Random set theory play important role in image processing, mathematical morphology, expert system, theoretical statistic, etc., see [15, 16, 23, 24, 25, 26, 45]. A sequence of probability measures $\{\mathsf{P}_n\}_{n\in\mathbf{N}}$ converges weakly to a probability measure P if $\int f d\mathsf{P}_n \to \int f d\mathsf{P}$ for all continuous, bounded real functions f. Several conditions equivalent to the weak convergence are provided by the theorem of Portmanteau ([3]). While the classical case considers weak convergence of sequences of probability measures, the main result presented here, see [14], is a theorem of Portmanteau type for sequences of capacity functionals for sequence of random closed sets.

The second recent application of the pseudo-analysis is related to the theory of Perona and Malik nonlinear partial differential equation, which is fundamental in image processing ([1, 5, 43]). In this model with partial differential equation a restored image can be seen as a version of the initial image at a special scale. We have proved for this equation the pseudo linear superposition principle [41].

2 Pseudo-operations and the general pseudo integral

Real operations used in this paper and non-additive measures are based on [2, 9, 18, 21, 33, 36]. Let $\leq$ be a total order on $[0,\infty]$.

Definition 1 *A binary operation* $\oplus : [0,\infty]^2 \to [0,\infty]$ *such that the following conditions are satisfied:*

(A1) $a \oplus b = b \oplus a$ (commutativity)

(A2) $a \leq a' \wedge b \leq b' \Rightarrow a \oplus b \leq a' \oplus b'$ (monotonicity)

(A3) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity)

(A4) $a \oplus 0 = a$ (neutral element)

(A5) $a_n \to a \wedge b_n \to b \Rightarrow (a_n \oplus b_n) \to a \oplus b$ (continuity)

is called pseudo-addition.

Definition 2 *For a given pseudo-addition* $\oplus$ *the corresponding* pseudo-multiplication *is a binary operation* $\odot : [0,\infty]^2 \to [0,\infty]$ *such that the following conditions are satisfied*

(M1) $a \odot b = b \odot a$ (commutativity)

(M2) $a \odot 0 = 0$ (zero element)

(M3) $a \leq a' \wedge b \leq b' \Rightarrow a \odot b \leq a' \odot b'$ (monotonicity)

(M4) $(a \oplus b) \odot c = (a \odot c) \oplus (a \odot c)$ (distributivity)

(M5) $a \odot \mathbf{1} = a$ (unit element)

(M6) $a \odot (b \odot c) = (a \odot b) \odot c$ (associativity)

(M7) $a_n \to a \wedge b_n \to b \Rightarrow (a_n \odot b_n) \to a \odot b$ (continuity)

The algebraic structure $([0,\infty], \oplus, \odot)$ is a semiring, see [11, 20]. Let $\mathcal{A}$ be a σ-algebra of subsets of non-empty abstract space Ω and $m : \mathcal{A} \to \mathbf{R}$ non-decreasing set function with $m(\emptyset) = 0$. Let $\mathcal{M}$ be a family of $\mathcal{A}$-measurable functions $f : \Omega \to [0,\infty]$. A measurable function $s : \Omega \to [0,\infty]$ is called a *simple function* if its range is finite. Let $Rang(s) = \{a_1, a_2, \ldots, a_k\}$ such that

$0 < a_1 < a_2 < \ldots < a_k$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. The *standard $\oplus$-step representation* of a simple function s is

$$s = \bigoplus_{i=1}^{k} b(c_i^*, C_i^*) \tag{1}$$

where $c_1^* = a_1,\ c_2^* = a_2 \ominus a_1,\ \ldots, c_m^* = a_m \ominus a_{m-1},\ C_i^* = \bigcup_{j=i}^{m} A_i$ and $b : \Omega \to [0, \infty]$ is a *basic function* given by

$$b(c_i^*, C_i^*)(\omega) = \begin{cases} c_i^*, & \omega \in C_i^*, \\ 0, & \omega \notin C_i^*. \end{cases}$$

Specially, for $c_i^* = \mathbf{1}$ and $A \in \mathcal{A}$, the basic function reduces to the *pseudo characteristic function* $\chi_A : \Omega \to \{0, \mathbf{1}\}$ given by

$$\chi_A(\omega) = \begin{cases} \mathbf{1}, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

Definition 3 *(a)* The pseudo integral of a simple function s *with the standard $\oplus$-step representation (1) is given by*

$$\int^{\oplus} s \odot\ dm = \bigoplus_{i=1}^{m} c_i^* \odot m(C_i^*).$$

(b) The pseudo integral *of a function $f \in \mathcal{M}$ is*

$$\int^{\oplus} f \odot\ dm = \sup\left\{ \int^{\oplus} s \odot\ dm : s \in \mathcal{S}_f \right\},$$

where $\mathcal{S}_f$ is a family of all simple function s such that $s \leq f$.

3 Random sets

For the purposes of presented generalization, the classical Lebesgue integral had been substituted with more general one, known as the pseudo integral ([2]). General pseudo integral is based on pseudo-addition and pseudo-multiplication, which are generalizations of the classical operations, and monotone set function (non-additive measure), see [33, 36].

We give some basic notions from theory of random sets ([23, 27, 28, 29, 30]). Collections of closed, open and compact subset of $\mathbf{R}$ are denoted with $\mathcal{F}$, $\mathcal{O}$ and $\mathcal{K}$, respectively. Of the special importance for the theory of random sets is the collection of closed sets $\mathcal{F}$, as well as its sub-collections $\mathcal{F}_G$, $G \in \mathcal{O}$ and $\mathcal{F}^K$, $K \in \mathcal{K}$ defined in the following way:

$$\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\}, \quad G \in \mathcal{O}$$

$$\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\}, \quad K \in \mathcal{K}.$$

Collections $\{\mathcal{F}_G : G \in \mathcal{O}\}$ and $\{\mathcal{F}^K : K \in \mathcal{K}\}$ generate a topology $\tau(\mathcal{F})$ on $\mathcal{F}$ (*hit-or-miss-topology*). $\mathcal{F}$ with the hit-or-miss topology is a compact, separable and Hausdorff space ([23]). Taking countable unions and intersections of the open sets of the topological space $(\mathcal{F}, \tau(\mathcal{F}))$, a σ-field $\Sigma(\mathcal{F})$ is generated in $\mathcal{F}$. We have by [23, 27, 28]

Definition 4 Random closed set S *is a measurable mapping from the probability space* $(\Omega, \mathcal{A}, \mathsf{P})$ *into the measurable space* $(\mathcal{F}, \Sigma(\mathcal{F}))$.

Random closed set S generates probability distribution $\mathbf{P}_\mathsf{S}$ in the following way

$$\mathbf{P}_\mathsf{S}(A) = \mathsf{P}(\{\omega \in \Omega : \mathsf{S}(\omega) \in A\}) = \mathbf{P}_\mathsf{S}(\mathsf{S} \in A), \text{ for all } A \in \Sigma(\mathcal{F}).$$

Probability distribution $\mathbf{P}_\mathsf{S}$ of the random closed set S is determined by functional T_S defined on the space of compact subsets of $\mathbf{R}$.

Definition 5 *The* capacity functional $\mathsf{T}_\mathsf{S} : \mathcal{K} \to [0,1]$ *of a random closed set* S *for* $K \in \mathcal{K}$ *is defined by*

$$\mathsf{T}_\mathsf{S}(K) = \mathbf{P}_\mathsf{S}(\mathsf{S} \in \mathcal{F}_K) = \mathbf{P}_\mathsf{S}(\mathsf{S} \cap K \neq \emptyset). \tag{2}$$

Although the capacity functional T_S given by (2) is defined only on $\mathcal{K}$, it can be extended onto the family $\mathcal{P}$ of all subsets of $\mathbf{R}$, in the following way

$$\mathsf{T}_\mathsf{S}^*(G) = \sup\{\mathsf{T}_\mathsf{S}(K) : K \in \mathcal{K},\ K \subset G\}, \ \ G \in \mathcal{G}$$

and

$$\mathsf{T}_\mathsf{S}^*(M) = \inf\{\mathsf{T}_\mathsf{S}^*(G) : G \in \mathcal{G},\ M \subset G\}, \ \ M \in \mathcal{P}.$$

A subset $M \subset \mathbf{R}$ is called *capacitable* if the following holds

$$\mathsf{T}_\mathsf{S}(M) = \sup\{\mathsf{T}_\mathsf{S}(K) : K \in \mathcal{K}, K \subset M\}.$$

Obviously, T_S^* coincides with T_S on $\mathcal{K}$, and all Borel sets B are capacitable ([23, 28]). We have by [28] the following theorem.

Theorem 6 *(i)* $\mathsf{T}_\mathsf{S}^*(K) = \mathsf{T}_\mathsf{S}(K)$ *for all* $K \in \mathcal{K}$.

(ii) For each Borel set B *it holds* $\mathsf{T}_\mathsf{S}^*(B) = \sup\{\mathsf{T}_\mathsf{S}(K) : K \in \mathcal{K}, K \subset B\}$.

Further on, $\mathsf{T}_\mathsf{S}^*(M)$ for all capacitable set M will be denoted with $\mathsf{T}_\mathsf{S}(M)$.

For a given random closed set S the corresponding capacity functional will be denoted by T. Capacity functionals of the random closed sets are uniquely characterized by the following proposition (see [23, 28, 29]).

Proposition 7 *(i) For all* $K \in \mathcal{K}$ *the inequality* $0 \leq \mathsf{T}(K) \leq 1$ *and* $\mathsf{T}(\emptyset) = 0$ *hold.*

(ii) For all random closed sets and for all $K_1, K_2 \in \mathcal{K}$ holds

$$K_1 \subseteq K_2 \Rightarrow \mathsf{T}(K_1) \leq \mathsf{T}(K_2).$$

(iii) The capacity functional is upper semi-continuous, i.e.,

$$K_n \downarrow K \Rightarrow \mathsf{T}(K_n) \downarrow \mathsf{T}(K), \quad \textit{for } K_n \in \mathcal{K}, \quad n = 1, 2, \ldots, \quad K \in \mathcal{K}.$$

We introduce the following generalization ([14]).

Definition 8 *A sequence of capacity functionals $\{\mathsf{T}_n\}_{n\in\mathbf{N}}$* $(\oplus, \odot)$-weak converges *(shortly,* pseudo-weak converges*) to capacity functional T if and only if for each continuous, bounded function $f : \mathbf{R} \to [0, \infty]$ holds*

$$\lim_{n\to\infty} \int^{\oplus} f \odot d\mathsf{T}_n = \int^{\oplus} f \odot d\mathsf{T}.$$

The following three theorems generalize the classical Portmantea-u theorem ([3]).

Theorem 9 *If a sequence of capacity functionals $\{\mathsf{T}_n\}_{n\in\mathbf{N}}$ pseudo-weak converges to capacity functional T, then*

$$\limsup_n \mathsf{T}_n(F) \leq \mathsf{T}(F)$$

for all closed sets $F \subseteq \mathbf{R}$.

Theorem 10 *If a sequence of capacity functionals $\{\mathsf{T}_n\}_{n\in\mathbf{N}}$ pseudo-weak converges to capacity functional T, then*

$$\liminf_n \mathsf{T}_n(G) \geq \mathsf{T}(G)$$

for all open set G.

Theorem 11 *If for all closed sets F holds $\limsup_n \mathsf{T}_n(F) \leq \mathsf{T}(F)$ and for all open sets G holds $\liminf_n \mathsf{T}_n(G) \geq \mathsf{T}(G)$, then sequence of capacity functionals $\{\mathsf{T}_n\}_{n\in\mathbf{N}}$ pseudo-weak converges to capacity functional T.*

4 Nonlinear partial differential equations

An image u is embedded in an evolution process, denoted by $u(t, \cdot)$. The original image is taken at time $t = 0$, $u(0, \cdot) = u_0(\cdot)$. The original image is then transformed, and this process can be written in the form $\frac{\partial u}{\partial t}(t, x) + F\left(x, u(t, x), \nabla u(t, x), \nabla^2 u(t, x)\right) = 0$ in Ω. Some possibilities for

F to restore an image are considered in [1]. PDE-methods for restoration is in general form:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t,x) + F\left(x, u(t,x), \nabla u(t,x), \nabla^2 u(t,x)\right) = 0 \text{ in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial N}(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \quad u(0,x) = u_0(x), \end{array} \right. \tag{3}$$

where $u(t,x)$ is the restored version of the initial degraded image $u_0(x)$. The idea is to construct a family of functions $\{u(t,x)\}_{t>0}$ representing successive versions of $u_0(x)$. As t increases $u(t,x)$ changes into a more and more simplified image. We would like to attain two goals. The first is that $u(t,x)$ should represent a smooth version of $u_0(x)$, where the noise has been removed. The second, is to be able to preserve some features such as edges, corners, which may be viewed as singularitis. The basic PDE in image restoration is the heat equation:

$$\left\{ \begin{array}{c} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0, t \geq 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x). \end{array} \right. \tag{4}$$

We consider that $u_0(x)$ is primarily defined on the square $[0,1]^2$. We extend it by symmetry to $C = [-1,1]^2$, and then on all $\mathbb{R}^2$, by periodicity. This way of extending $u_0(x)$ is classical in image processing. If $u_0(x)$ is extended in this way and satisfies in addition $\int_C |u_0(x)|\,dx < +\infty$, we will say that $u_0 \in L^1_{\#}(C)$ (see [1]). Solving (4) is equivalent to carrying out a Gaussian linear filtering, which was widely used in signal processing. If $u_0 \in L^1_{\#}(C)$, then the explicit solution of (4) is given by

$$u(t,x) = \int_{\mathbb{R}^2} G_{\sqrt{2t}}(x-y)\, u_0(y)\, dy = \left(G_{\sqrt{2t}} * u_0\right)(x),$$

where $G_\sigma(x)$ denotes the two-dimensional Gaussian kernel

$$G_\sigma(x) = \frac{1}{2\pi\sigma} e^{-\frac{|x|^2}{2\sigma^2}}$$

The heat equation has been (and is) successfully applied in image processing but it has some drawback. It is too smoothing and because of that edges can be lost or severely blurred. In [1] authors consider models that are generalizations

of the heat equation. The domain image will be a bounded open set Ω of $\mathbb{R}^2$. The following equation is initially proposed by Perona and Malik [43]:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = \operatorname{div}\left(c\left(|\nabla u|^2\right)\nabla u\right) \quad \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial N} = 0 \text{ on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega \end{array} \right. \tag{5}$$

where $c : [0,\infty) \to (0,\infty)$. If we choose $c \equiv 1$, then it is reduced on the heat equation. If we assume that $c(s)$ is a decreasing function satisfying $c(0) = 1$ and $\lim_{s\to\infty} c(s) = 0$, then inside the regions where the magnitude of the gradient of u is weak, equation (5) acts like the heat equation and the edges are preserved. For each point x where $|\nabla u| \neq 0$ we can define the vectors $N = \frac{\nabla u}{|\nabla u|}$ and T with $T \cdot N = 0$, $|T| = 1$. For the first and second partial derivatives of u we use the usual notation u_{x_1}, u_{x_2}, $u_{x_1 x_1}$,... We denote by u_{NN} and u_{TT} the second derivatives of u in the T-direction and N-direction, respectively:

$$\begin{aligned} u_{TT} &= T^t \nabla^2 u T = \frac{1}{|\nabla u|^2} \left(u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy} \right), \\ u_{NN} &= N^t \nabla^2 u N = \frac{1}{|\nabla u|^2} (u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}). \end{aligned}$$

The first equation in (5) can be written as

$$\frac{\partial u}{\partial t}(t,x) = c\left(|\nabla u(t,x)|^2\right) u_{TT} + b\left(|\nabla u(t,x)|^2\right) u_{NN}, \tag{6}$$

where $b(s) = c(s) + 2sc'(s)$. Therefore, (6) is a sum of a diffusion in the T-direction and a diffusion in the N-direction. The function c and b act as weighting coefficients. Since N is normal to the edges, it would be preferable to smooth more in the tangential direction T than in the normal direction. Because of that we impose

$$\lim_{s \to \infty} \frac{b(s)}{c(s)} = 0 \quad \text{or} \quad \lim_{s \to \infty} \frac{sc'(s)}{c(s)} = -\frac{1}{2} \tag{7}$$

If $c(s) > 0$ with power growth, then (7) implies that $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$. The equation (5) is parabolic if $b(s) > 0$. The assumptions imposed on $c(s)$ are

$$\begin{cases} c : [0,\infty) \to (0,\infty) \ \text{decreasing}, \\ c(0) = 1, \ c(s) \approx \frac{1}{\sqrt{s}} \ \text{as } s \to \infty, \\ b(s) = c(s) + 2sc'(s) > 0. \end{cases} \tag{8}$$

Often used function $c(s)$ satisfying (8) is $c(s) = \frac{1}{\sqrt{1+s}}$. Because of the behavior $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$, it is not possible to apply general results from parabolic equations theory. Framework to study this equation is nonlinear semigroup theory (see [1, 4, 6]).

We have proved in [41] that the pseudo-linear superposition principle holds for Perona and Malik equation.

Theorem 12 *If $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are solutions of the equation*

$$\frac{\partial u}{\partial t} - div\left(c\left(|\nabla u|^2\right)\nabla u\right) = 0, \tag{9}$$

then $u_1 \oplus u_2$ is also a solution of (9) on the set

$$D = \{(t,x)\, | t \in (0,T), x \in \mathbb{R}^2, u_1(t,x) \neq u_2(t,x)\},$$

with respect to the operation $\oplus = \min$.

The obtained results will serve for further investigation of the weak solutions of the equation (9) in the sense of Maslov [19, 22, 36] and Gondran [12, 13], as well as their important applications.

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