Integrals based on non-additive measures

Endre Pap

Department of Mathematics and Informatics, University of Novi Sad
Trg D. Obradovića 4, 21 000 Novi Sad, Serbia
e-mail: pape@eunet.yu

Abstract: There are presented two recent results on integrals based on non-additive measures. First is related to Jensen type inequality for a pseudo-integral, and the second is a connection of integral with aggregation functions with infinite inputs.

Keywords: Integral, pseudo-addition, pseudo-multiplication, capacity, Choquet integral.

1 Introduction

In this paper we present two recent applications of the theory of integrals based on non-additive measures.

In [19] it was proven a Jensen type inequality for the Sugeno integral and authors analyze the necessary conditions for the reverse Jensen's inequality. In this paper we show a Jensen type inequality for the pseudo-integral, in Section 2. Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers in [7, 9, 13, 14, 15] a general semiring is defined on a real interval.

Aggregation of countably infinitely many inputs occurs in applications, as decision problems with an infinite jury, game theory with infinitely many players, etc. They enable a better understanding of decision problems with extremely huge juries, game theoretical problems with extremely many players, etc., see [12, 17, 21]. In Section 3 we present some recent results on aggregation functions with infinitely many inputs.

2 Jensen type inequality for pseudo-integral

Let $[a, b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty, \infty]$. We consider here the total order $\leq$ on $[a, b]$. The operation $\oplus$ (pseudo-addition) is a commutative, non-decreasing, associative function $\oplus : [a, b] \times [a, b] \to [a, b]$ with a zero (neutral) element denoted by $\mathbf{0}$. Denote $[a, b]_+ = \{x : x \in [a, b], x \geq \mathbf{0}\}$. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \leq y$ implies $x \odot z \leq$

$y \odot z, z \in [a,b]_+$,associative and for which there exist a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $1 \odot x = x$. We assume $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a,b], \oplus, \odot)$ is called a *semiring* (see [8, 13]). We suppose further that the operations $\oplus$ and $\odot$ are continuous.

Let X be a non-empty set. Let $\mathcal{A}$ be a σ-algebra of subsets of X. A set function $m : \mathcal{A} \to [a,b]_+$ (or semiclosed interval) is a $\oplus$- measure if there hold $m(\varnothing) = \mathbf{0}$ (if $\oplus$ is not idempotent), and m is σ-$\oplus$-(decomposable) measure, i.e., $m(\cup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$ holds for any sequence $(A_i)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$. The characteristic function with values in a semiring is defined with

$$\chi_A(x) = \begin{cases} \mathbf{0} & , \quad x \notin A \\ \mathbf{1} & , \quad x \in A \end{cases} .$$

An elementary (measurable) function is mapping $e : X \to [a,b]$ that has the following representation $e = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i}$ for $a_i \in [a,b]$ and sets $A_i \in \mathcal{A}$ disjoint if $\oplus$ is nonidempotent. The pseudo-integral of a bounded measurable function $f : X \to [a,b]$, (for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$ there exists a monotone ε-net in $f(X)$) is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} e_n(x) \odot dm,$$

where $(e_n)_{n\in\mathbb{N}}$ is a sequence of elementary functions which converges uniformly to f.

We shall consider the semiring with pseudo-operations for two completely different cases.

The first case is when pseudo-operations are defined by a monotone and continuous function $g : [a,b] \to [0,\infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x) \cdot g(y)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.

The pseudo-integral reduces on g-integral, i.e.,

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))\,dx\right).$$

The second case is when the semiring is of the form $([a,b], \max, \odot)$, i.e., pseudo-addition is idempotent, and the pseudo-multiplication not. Here pseudo-integral is given with

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x\in\mathbb{R}} \left(f(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure m.

Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition ([10]).

The well-known Jensen inequality is a part of the classical mathematical analysis.

Theorem 1 *Let h be real and integrable function on $[0,1]$, $a < h(x) < b$, $x \in [0,1]$ and φ a convex function on (a,b). Then*

$$\varphi\left(\int_0^1 h(x)dx\right) \leqslant \int_0^1 \varphi(h(x))\,dx.$$

We have proved in [16] the following generalization of Jensen inequality.

Theorem 2 *Let $\Phi : [a,b] \to [a,b]$ be a convex and nonincreasing function. If an additive generator $g : [a,b] \to [a,b]$ of the pseudo-adition $\oplus$ is a convex and increasing function, then for any measurable function $f : [0,1] \to [a,b]$ holds:*

$$\Phi\left(\int_{[0,1]}^{\oplus} f(x)dx\right) \leq \int_{[0,1]}^{\oplus} \Phi(f(x))\,dx. \tag{1}$$

Example 3 *(i) Let $g(x) = x^{\alpha}$ for some $\alpha \in [1,\infty)$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^{\alpha}+y^{\alpha}}$ and $x \odot y = xy$. Then (1) reduces on the following inequality*

$$\Phi\left(\sqrt[\alpha]{\int_{[0,1]} f(x)^{\alpha}\,dx}\right) \leqslant \sqrt[\alpha]{\int_{[0,1]} \Phi(f(x))^{\alpha}\,dx}.$$

(ii) Let $g(x) = e^x$. The corresponding pseudo-operations are $x \oplus y = \log(e^x + e^y)$ and $x \odot y = x + y$. Then (1) reduces on the following inequality

$$\Phi\left(\ln \int_{[0,1]} e^{f(x)}\,dx\right) \leqslant \ln\left(\int_{[0,1]} e^{\Phi(f(x))\,dx}\right).$$

Now we consider the second case, when $\oplus = \max$, and $\odot = g^{-1}(g(x)g(y))$. Using the result from [10] there was proved in [16] the following generalization of the Jensen inequality.

Theorem 4 *Let $\Phi : [a,b] \to [a,b]$ be a convex and nonincreasing function, and $\odot$ is represented by a convex and increasing multiplicative generator g. Then for any continuous function $f : [0,1] \to [a,b]$ holds:*

$$\Phi\left(\int_{[0,1]}^{\sup} f \odot dm\right) \leqslant \int_{[0,1]}^{\sup} \Phi(f) \odot dm.$$

Example 5 *Using Example 3(ii) we have that $g^{\lambda}(x) = e^{\lambda x}$. Then*

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} + e^{\lambda y}\right) = \max(x,y),$$

and

$$x \odot_{\lambda} y = x + y.$$

Therefore Jensen type inequality from Theorem 4 reduces on

$$\Phi\left(\sup(f(x) + \psi(x))\right) \leqslant \sup\left(\Phi(f(x)) + \psi(x)\right),$$

where ψ defines sup*-measure m.*

3 Infinite aggregation functions

In this section, based on [3, 11], we present infinitary aggregation functions on sequences possessing some a priori given properties, and we give the connection with Choquet integral. We consider the set $[0,1]^{\mathbb{N}}$ of all sequences $\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots)$, where $x_i \in [0,1]\,(i \in \mathbb{N})$. The input space $[0,1]^{\mathbb{N}}$ equipped with standard Cartesian ordering, (i.e., $\mathbf{x} \leqslant \mathbf{y}$ means $x_i \leqslant y_i, i \in \mathbb{N}$), is a complete lattice with bottom element $\mathbf{0} = \{0\}^{\mathbb{N}}$ and top element $\mathbf{1} = \{1\}^{\mathbb{N}}$. We equip $[0,1]^{\mathbb{N}}$ with coordinatewise convergence, i.e., a sequence $\mathbf{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_i^{(n)}, \ldots)$ from $[0,1]^{\mathbb{N}}$ converges to $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots) \in [0,1]^{\mathbb{N}}$ if and only if $\lim_{n\to\infty} x_i^{(n)} = x_i$ for all $i \in \mathbb{N}$.

Definition 6 *A function $\mathsf{A}^{(\infty)} \colon [0,1]^{\mathbb{N}} \to [0,1]$ is an (infinitary) aggregation function if it satisfies the following conditions:*

(i) nondecreasing monotonicity, i.e., $\mathbf{x} \leqslant \mathbf{y}$ implies $\mathsf{A}^{(\infty)}(\mathbf{x}) \leqslant \mathsf{A}^{(\infty)}(\mathbf{y})$.

(ii) $\mathsf{A}^{(\infty)}(\mathbf{0}) = 0$ and $\mathsf{A}^{(\infty)}(\mathbf{1}) = 1$.

Properties of these functions are defined similarly to the corresponding properties of n-ary aggregation functions ([1, 3, 20]).

Additivity of the aggregation function implies its comonotone additivity, which yields its idempotence. On the other hand, there are no aggregation functions $\mathsf{A}^{(\infty)} \colon [0,1]^{\mathbb{N}} \to [0,1]$ which are both additive and symmetric.

Proposition 7 *An additive function* $\mathsf{F} : [0,1]^{\mathbb{N}} \to [0,1]$ *is homogeneous and nondecreasing. If* F *satisfies additionally* $\mathsf{F}(\mathbf{0}) = 0$ *and* $\mathsf{F}(\mathbf{1}) = 1$*, then it is an (infinitary) aggregation function.*

Corollary 8 *An aggregation function* $\mathsf{A}^{(\infty)}\colon [0,1]^{\mathbb{N}} \to [0,1]$ *is additive and continuous if and only if* $\mathsf{A}^{(\infty)}(\mathbf{x}) = \sum_{n=1}^{\infty} w_n x_n$ *for all* $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$*, where* $(w_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$*,* $\sum_{n=1}^{\infty} w_n = 1$*.*

Remark 9 *The arithmetic mean* $\mathsf{AM}^{(n)}\colon [0,1]^n \to [0,1]$ *is characterized as the unique* n*-ary additive symmetric aggregation function. Symmetry forces the equality of weights* $w_1 = \cdots = w_n = \frac{1}{n}$*. However, requiring similar properties on* $[0,1]^{\mathbb{N}}$ *can be reduced to looking for a sequence of weights* $(w_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ *such that all weights are equal and* $\sum_{n=1}^{\infty} w_n = 1$*. Evidently, such a sequence of weights cannot exist.*

The next result is derived from [2], see also [13].

Proposition 10 *An aggregation function* $\mathsf{A}^{(\infty)}\colon [0,1]^{\mathbb{N}} \to [0,1]$ *is comonotonic additive and lower semicontinuous if and only if there is a lower semicontinuous capacity (fuzzy measure)* $m\colon 2^{\mathbb{N}} \to [0,1]$ *(for each nondecreasing sequence* $(A_n)_{n\in\mathbb{N}}$ *in* $2^{\mathbb{N}}$ *and for each* $A \in 2^{\mathbb{N}}$*, with* $(A_n)_{n\in\mathbb{N}}$ *increasing to* A*, we have* $\lim_{n\to\infty} m(A_n) = m\left(\cup_{n\in\mathbb{N}} A_n\right)$*), such that*

$$\mathsf{A}^{(\infty)}(\mathbf{x}) = (C)\int_{\mathbb{N}} \mathbf{x}\,\mathrm{d}m = \int_0^1 m(\{i \in \mathbb{N} \mid x_i \geqslant t\})\,\mathrm{d}t, \tag{2}$$

i.e., $\mathsf{A}^{(\infty)}$ *is the Choquet integral with respect to* m*. Note that for any* $E \subset \mathbb{N}$ *we then have* $m(E) = \mathsf{A}(\mathbf{1}_E)$*.*

The symmetry of $\mathsf{A}^{(\infty)}\colon [0,1]^{\mathbb{N}} \to [0,1]$ when it is a Choquet integral-based aggregation function is related to the symmetry of the corresponding capacity $m\colon 2^{\mathbb{N}} \to [0,1]$, i.e.,

$$m(A) = m(\{\sigma(n) \mid n \in A\})$$

for all $A \subset \mathbb{N}$ and any bijective mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$. The symmetric capacity play important role in the characterization of infinitary OWA operator [18] (for finite OWA see [22]).

Definition 11 *A comonotone additive symmetric aggregation function* $\mathsf{A}^{(\infty)} : [0,1]^{\mathbb{N}} \to [0,1]$ *is called an infinitary OWA operator.*

Theorem 12 *A mapping* $\mathsf{A}^{(\infty)} : [0,1]^{\mathbb{N}} \to [0,1]$ *is an infinitary OWA operator if and only if there exists a symmetric measure* $m : 2^{\mathbb{N}} \to [0,1]$ *such that (2) holds.*

For a given extended aggregation function $\mathsf{A}\colon \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$, we look for an appropriate aggregation function $\mathsf{A}^{(\infty)}\colon [0,1]^{\mathbb{N}} \to [0,1]$ somehow linked to A. A natural approach is to define $\mathsf{A}^{(\infty)}$ as a limit of $(\mathsf{A}^{(n)})_{n\in\mathbb{N}}$,

$$\mathsf{A}^{(\infty)}((x_n)_{n\in\mathbb{N}}) := \lim_{n\to\infty} \mathsf{A}^{(n)}(x_1, \ldots, x_n). \tag{3}$$

If this limit exists, for any $(x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, we accept $\mathsf{A}^{(\infty)}$ given by (3) as an extension of A to the domain $[0,1]^{\mathbb{N}}$, and we keep the notation A also for $\mathsf{A}^{(\infty)}$ whenever appropriate. The aggregation function A is called *countably extendable*.

Definition 13 *An extended aggregation function* $\mathsf{A}\colon \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *is said to have a downwards (respectively, an upwards) attitude whenever, for any* $n \in \mathbb{N}$ *and any* $x_1, \ldots, x_{n+1} \in [0,1]$, *we have* $\mathsf{A}(x_1, \ldots, x_n, x_{n+1}) \leqslant \mathsf{A}(x_1, \ldots, x_n)$ *(respectively,* $\mathsf{A}(x_1, \ldots, x_n, x_{n+1}) \geqslant \mathsf{A}(x_1, \ldots, x_n)$*).*

Proposition 14 *(i) Each downwards (respectively, upwards) extended aggregation function* $\mathsf{A}\colon \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *is countably extendable.*

(ii) Let $\mathsf{T}\colon \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *(respectively,* $\mathsf{S}\colon \cup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$*) be an extended t-norm (respectively, extended t-conorm). Then* T *(respectively,* S*) is countably extendable.*

Acknowledgment

The author would like to thank for the support in part by the project MNZŽSS 144012, grant of MTA of HTMT, Slovak-Serbian bilateral project, and by the project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" of Academy of Sciences and Arts of Vojvodina supported by Provincial Secretariat for Science and Technological Development of Vojvodina.

References

[1] Beliakov, G.,Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners, Studies in Fuziness and Soft Computing, Springer, Berlin, 2007.

[2] P. Benvenuti, R. Mesiar, D. Vivona, *Monotone Set Functions-Based Integrals*, in Handbook of Measure Theory (Ed. E. Pap), Volume II, Elsevier, North-Holland, (2002), 205–232.

[3] Grabisch, M., Marichal, J. L., Mesiar, R., Pap, E.: Aggregation Functions, Cambridge University Press (in press).

[4] González, L., Muel, E., Mesiar, R.: What is the arithmetic mean of an infinite sequence? Proc. ESTYLF'2002, Leon, 2002, 183–187.

[5] González, L., Muel, E., Mesiar, R.: A remark on the arithmetic mean of an infinite sequence. Internat. J. Uncertainty, Fuzziness, Knowledge-Based Systems 10, Suppl. (2002), 51–58.

[6] E. P. Klement, R. Mesiar, E. Pap, *Triangular Norms,* Kluwer, Dordrecht, 2000.

[7] V. N. Kolokoltsov, V.P. Maslov, *Idempotent Analysis and Its Applications,* Kluwer, Dordrecht, 1997.

[8] W. Kuich, *Semirings, Automata, Languages,* Berlin, Springer-Verlag, 1986.

[9] V. P. Maslov, S. N. Samborskij (eds.), *Idempotent Analysis,* Advances in Soviet Mathematics 13, Providence, Rhode Island, Amer. Math. Soc., 1992.

[10] R. Mesiar, E. Pap, Idempotent integral as limit of g-integrals, Fuzzy Sets and Systems 102 (1999), 385-392.

[11] R. Mesiar, E. Pap: Aggregation of infinite sequences, Information Sciences 178(18) (2008), 3557-3564.

[12] Neyman, A.: Values of games with infinitely many players. In: R.J. Aumann, S. Hart (ed.), 2002. "Handbook of Game Theory with Economic Applications," Handbook of Game Theory with Economic Applications, Elsevier, edition 1, volume 3, number 3, chapter 56, 2121-2167.

[13] E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publisher, Dordrecht, 1995.

[14] E. Pap, Pseudo-analysis as a mathematical base for soft computing, *Soft Computing* 1 (1997), 61-68.

[15] E. Pap, Pseudo-additive measures and their applications, in Handbook of Measure Theory (Ed. E. Pap), North-Holland, Elsevier, Amsterdam, 2002, 1403-1468.

[16] E. Pap, M. Štrboja, Generalization of the Jensen's Inequality for Pseudo-Integral, Proceedings of the 6th International Symposium on Intelligent Systems and Informatics,SISY 2008, IEEE 1-4244-2407-8/08/, Subotica, 2008.

[17] Rovatti, R., Fantuzzi, C.: s-norm aggregation of infinite collections. Fuzzy Sets and Systems 84 (1996), 255-269.

[18] Stupňanova, A.: Infinitary OWA operators, International Conference 70 Years of FCE STU, December 4-5, 2008, Bratislava, Slovakia.

[19] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, A Jensen type inequality for fuzzy integrals, Information Science 177 (2007), 3192-3201.

[20] Torra, V., Narukawa, Y.: Modeling decisions: Information Fusion and Aggregation Operators, Cognitive Technologies, Springer, 2007.

[21] Vallentyne, P., Kagan, Sh.: Infinite Value and Finitely Additive Value Theory. Journal of Philosophy 94 (1997): 5-26.

[22] Yager, R. R.: On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. Syst., Man, Cybern. 18 (1988), 183–190