Preferences and Decisions without Numbers*

János Fodor

Dept. of Biomathematics and Informatics, Faculty of Veterinary Science
Szent István University, István u. 2, H-1078 Budapest, Hungary
E-mail: Fodor.Janos@aotk.szie.hu

Abstract: We extend a multiple expert, multiple criteria decision aiding technique on an ordinal evaluation scale, originally proposed by Yager [12]. This extension is heavily based on the definition of logic and averaging operators on such scales. We can show a variety of these operations, so the decision maker can have some freedom in choosing the most appropriate one for his/her special purpose. In particular, the case of smooth underlying operators (t-norms and t-conorms) is handled with special emphasis. The applicability of the results is illustrated by an example.

Keywords: Individual preferences, multiple criteria decision aid, linguistic scale, implication and averaging operators.

1 Introduction

Complex decision situations, when multiple experts take part in evaluation of alternatives and such an evaluation is based on multiple criteria, are rather difficult to be handled. In such processes, one of the main issues is to obtain a reasonable aggregation of experts' evaluation in order to get a global score of each alternative by each expert. Next, we want to have an overall score for each alternative to be able to choose one or more "best" of them.

One of the usual evaluation scales is a subset of real numbers (e.g. a compact interval such as $[0,1]$), with its rich algebraic structure. However, Yager [12] emphasized an undesirable effect of such a numeric scale called *the tyranny of numbers*: " ... the numbers take a life and precision far in excess of the ability of the evaluators in providing these scores". At the same time, Yager suggested an approach of using a finite ordinal scale for the evaluation instead of a subset of real numbers. Such an ordinal scale can represent simple linguistic terms like *None, Very Low, Low, Medium, High, Very High, Perfect*, in accordance with the observation of psychologists that "human beings can reasonably manage to keep in mind seven or so items" (Yager ([12], Miller [10]). We will use this typical scale S

$$S = \{\mathrm{N}, \mathrm{VL}, \mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}, \mathrm{P}\}, \tag{1}$$

* This research has been supported in part by OTKA T046762, and by the Bilateral Scientific and Technological Cooperation Flanders-Hungary BIL00/51 (B-08/2000).

where the letters refer to the previous linguistic terms, and they are listed in an increasing order.

To define reasonable aggregation functions on an ordinal scale, we employ an idea of Fodor and Roubens [4], and also the original approach of Yager [12]. The combination of these two sources will provide us a variety of acceptable aggregation functions which can be used in complex decision problems. For a similar approach to discrete preference structures we refer to De Baets and Fodor [1]

2 Problem Formulation

We denote by $\mathbb{N}$ the set of positive integers. For any $\ell \in \mathbb{N}$, let K_ℓ be the set of integers from 1 to ℓ; that is, $K_\ell = \{1, 2, \ldots, \ell\}$.

Let $m, n, q, r \in \mathbb{N}$, $A = \{a_1, a_2, \ldots, a_n\}$ be a set of *alternatives*, $E = \{e_1, e_2, \ldots, e_r\}$ be a group of *experts* (evaluators), $S = \{s_1, s_2, \ldots, s_q\}$ be an *ordinal evaluation scale* with the linear ordering $s_i \prec s_j \iff i < j$, $(i, j \in K_q)$, $C = \{c_1, c_2, \ldots, c_m\}$ be given *criteria* for the evaluation of the alternatives. Denote $P_{ik}(c_j) \in S$ the *rating* of the ith alternative on the jth criterion by the kth expert, and $w(c_j) \in S$ the *importance* (weight) associated with criterion c_j $(i \in K_n,\ j \in K_m,\ k \in K_r)$.

First, we want to determine the global score P_{ik} of each alternative a_i by each expert e_k $(i \in K_n,\ k \in K_r)$. That is, we are looking for a function $F : S^m \times S^m \to S$ such that $P_{ik} = F(P_{ik}(c_1), \ldots, P_{ik}(c_m); w(c_1), \ldots w(c_m))$ for $i \in K_m,\ k \in K_r$. Such a function F can be interpreted as an *averaging* operator. Typically, if the evaluation scale is a subset of real numbers then F is chosen as the weighted arithmetic mean. Because in an ordinal scale we do not have operations like addition, the main difficulty is to find appropriate forms of F.

Second, after having the values P_{ik}, we want to combine experts' opinion in order to find the overall evaluation of alternatives.

3 Smooth De Morgan Triplets on Finite Chains

Since continuous operations play a key role when the evaluation scale is a compact interval, in this section we briefly summarize the corresponding notion (called *smoothness*) on a finite totally ordered evaluation scale (like S in (1) above). In this section we use notation different from the rest of the paper, to be able to present these results in the spirit of their generality.

Assume that $\mathcal{L} = \{x_0, x_1, \ldots, x_n, x_{n+1}\}$ is a totally ordered finite set of $n + 2$ elements that are indexed increasingly: $x_0 \prec x_1 \prec \ldots \prec x_n \prec x_{n+1}$, according to a relation $\prec$. For $x \in \mathcal{L}$, let $\mathrm{ind}(x)$ denote the number of elements y such that $y \prec x$.

In the sequel, we use the notations $\mathbf{0} = x_0$ and $\mathbf{1} = x_{n+1}$. For any $x_i, x_j \in \mathcal{L}$ such that $x_i \preceq x_j$, let us define $\langle x_i, x_j \rangle = \{x_k \in \mathcal{L} \mid x_i \preceq x_k \preceq x_j\}$, which can be considered as the discrete "closed interval" of points in $\mathcal{L}$ between x_i and x_j (note that $x \preceq y$ if and only if either $x \prec y$, or $x = y$).

3.1 Strong Negations

First we consider *strong negations* on $\mathcal{L}$. That is, decreasing functions $\mathcal{N} : \mathcal{L} \to \mathcal{L}$ with $\mathcal{N}(\mathbf{0}) = \mathbf{1}$ that are *involutive*: $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in \mathcal{L}$.

Theorem 1. [9] *The unique strong negation $\mathcal{N}$ on $\mathcal{L} = \{x_0, \ldots, x_{n+1}\}$ is given by*

$$\mathcal{N}(x_i) = x_{n-i+1} \ ,$$

for all $x_i \in \mathcal{L}$.

Clearly, this strong negation on $\mathcal{L}$ corresponds to the standard negation $x \mapsto 1 - x$ on the unit interval. Notice however that $\mathcal{N}$ does not have a fixed point (i.e. an $x \in \mathcal{L}$ such that $\mathcal{N}(x) = x$) when n is even.

3.2 Smooth t-norms and t-conorms

Mayor and Torrens [9] have determined all associative, commutative, increasing binary operations $\mathcal{T} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ that satisfy $\mathcal{T}(\mathbf{1}, \mathbf{1}) = \mathbf{1}$, and for all $x, y \in \mathcal{L}$

$$x \preceq y \iff (\exists \in \mathcal{L})(x = \mathcal{T}(y, z)) \ . \tag{2}$$

Such a binary operation $\mathcal{T}$ is called a *smooth t-norm on* $\mathcal{L}$. Indeed, it can be seen that $\mathcal{T}$ satisfies all the four axioms of t-norms, and condition (2) is just equivalent with the continuity of $\mathcal{T}$ when it is considered on $[0, 1]$. For more details on the smoothness property of binary operations, we refer to [8,6,7]. A smooth t-norm on $\mathcal{L}$ is called *Archimedean* if $\mathcal{T}(x, x) \prec x$ for all $x \in \langle x_1, x_n \rangle$ [9].

Theorem 2. [9] *The only Archimedean smooth t-norm on $\mathcal{L} = \{x_0, \ldots, x_{n+1}\}$ is given by*

$$\mathcal{T}(x_i, x_j) = \begin{cases} x_{i+j-(n+1)} & , \textit{if } i + j > n + 1 \\ \mathbf{0} & , \textit{otherwise} \end{cases} \ , \tag{3}$$

for all $x_i, x_j \in \mathcal{L}$.

One recognizes that the above t-norm corresponds to the Łukasiewicz t-norm on $[0, 1]$; hence, it is denoted by $\mathcal{T}_L$ in the sequel. As a consequence, one essential difference between the finite case and the usual unit interval is that there exists only one Archimedean smooth t-norm on a given finite $\mathcal{L}$, and it depends basically on the cardinality (i.e., the number of elements) of $\mathcal{L}$. No counterpart of any strict t-norm (like the product on $[0, 1]$) exists on $\mathcal{L}$.

If, in the remaining case, $\mathcal{T}$ is a non-Archimedean smooth t-norm then it has idempotent(s) other than $\mathbf{0}$ and $\mathbf{1}$: $\exists\ x_i \in \langle x_1, x_n\rangle$ such that $T(x_i, x_i) = x_i$. Then $\mathcal{T}(x_k, x_\ell) = x_{\min\{k,\ell\}}$ if there exists an idempotent x_i between x_k and x_ℓ. In any other case, there are two consecutive idempotents x_i, x_j such that $x_i \preceq x_k, x_\ell \preceq x_j$, and

$$\mathcal{T}(x_k, x_\ell) = \begin{cases} x_i & \text{, if } \ell + k \leq i + j \\ x_{\ell+k-i} & \text{, otherwise} \end{cases} .$$

Denote by $\mathcal{T}_M$ the smooth t-norm on $\mathcal{L}$ which corresponds to the minimum; it is the only t-norm that has all elements as idempotents. Moreover, there are exactly 2^n different smooth t-norms on $\mathcal{L}$ (where $|\mathcal{L}| = n + 2$) [9].

Smooth t-conorms can be obtained by duality w.r.t. the unique strong negation $\mathcal{N}$. Obviously, the only Archimedean smooth t-conorm can be obtained from Eq. (3) as follows:

$$\mathcal{S}_L(x_i, x_j) = \begin{cases} x_{i+j} & \text{, if } i + j < n + 1 \\ \mathbf{1} & \text{, otherwise} \end{cases} , \tag{4}$$

for all $x_i, x_j \in \mathcal{L}$.

A triplet $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ on $\mathcal{L}$ is called a *smooth De Morgan triplet* if $\mathcal{T}$ is a smooth t-norm on $\mathcal{L}$, $\mathcal{N}$ is the unique strong negation on $\mathcal{L}$ and $\mathcal{S}(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$ is the dual of $\mathcal{T}$. Obviously, such a De Morgan triplet depends *only* on the choice of the smooth t-norm $\mathcal{T}$. The triplet $L = (\mathcal{T}_L, \mathcal{S}_L, \mathcal{N})$ is called *the* Łukasiewicz triplet.

4 Scores of the Alternatives

In this section the methodology proposed by Yager [11] is extended. Our goal is to find the score of each alternative by each expert. Intuitively, the following principles seems to be acceptable in order to find the aggregation function:

- criteria having low importance should have little effect on the overall score;
- if a criterion is important then it must have a good score;
- the previous statement is applied to all criteria.

Based on such assumptions, Yager [11] proposed the following way of aggregation ($i \in K_n,\ k \in K_r$):

$$P_{ik} = \bigwedge_{j=1}^{m} \left[\mathcal{N}(w(c_j)) \vee P_{ik}(c_j)\right], \tag{5}$$

where $\mathcal{N} : S \to S$ is the standard negation defined on S (see Theorem 1 above).

If we look at the formula in (5) then we can verify the above principles.

From a formal point of view, the operation $\mathcal{N}(x) \vee y$ is nothing else but an *implication* on S $(x, y \in S)$. Updating Definition 1.15 from Fodor and Roubens [4], a function $\mathcal{I} : S^2 \to S$ is called an *implication* on S if and only if $\mathcal{I}$ satisfies the following conditions:

1. If $x \prec z$ then $\mathcal{I}(x, y) \succ \mathcal{I}(z, y)$ for all $x, y, z \in S$.
2. If $y \prec t$ then $\mathcal{I}(x, y) \prec \mathcal{I}(x, t)$ for all $x, y, t \in S$.
3. $\mathcal{I}(\mathbf{0}, x) = \mathbf{1}$ for all $x \in S$.
4. $\mathcal{I}(x, \mathbf{1}) = \mathbf{1}$ for all $x \in S$.
5. $\mathcal{I}(\mathbf{1}, \mathbf{0}) = \mathbf{0}$.

Now we can extend (5) by using an idea of Fodor and Roubens [4]. Two classes of idempotent and monotonic aggregation operations were suggested on the closed unit interval (see Proposition 5.4 in the cited book). According to those formulas, we can define the corresponding aggregations when only a linear scale S is considered.

Suppose that $\mathcal{I}$ is an implication on S satisfying the following additional conditions:

6. $\mathcal{I}(\mathbf{1}, x) = x$ for any $x \in S$.
7. $\mathcal{I}(x, y) \succeq y$ for all $x, y \in S$.

Moreover, let $\mathcal{N}$ be the standard negation on S. Consider the following two classes of operations on S:

$$\mathcal{M}_\wedge(t_1, \ldots, t_\ell) = \bigwedge_{i=1}^{\ell} \mathcal{I}(w_i, t_i),$$

$$\mathcal{M}_\vee(t_1, \ldots, t_\ell) = \bigvee_{i=1}^{\ell} \mathcal{N}(\mathcal{I}(w_i, \mathcal{N}(t_i)),$$

where $w_1, \ldots, w_\ell \in S$ are weights such that $\bigvee_{i=1}^{\ell} w_i = \mathbf{1}$. Under these conditions, thus defined $\mathcal{M}_\wedge$ and $\mathcal{M}_\vee$ are idempotent, nondecreasing functions. In fact, $\mathcal{M}_\wedge$ can be used in (5) in order to define the scores P_{ik} $(i \in K_n,\ k \in K_r)$:

$$P_{ik} = \bigwedge_{j=1}^{m} \mathcal{I}(w(c_j), P_{ik}(c_j)). \tag{6}$$

Now we list some implication functions that satisfy conditions 1-7 above. Note that all are induced by well-known fuzzy implications defined on $[0, 1]$. The original formula (5) corresponds to $\mathcal{I}_1$.

$$\mathcal{I}_1(s_i, s_j) = s_{\max(q-i+1,j)},$$

$$\mathcal{I}_2(s_i, s_j) = \begin{cases} \mathbf{1} & \text{if } i \le j \\ s_{\max(q-i+1,j)} & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_3(s_i, s_j) = s_{\min(q-i+j,q)},$$

$$\mathcal{I}_4(s_i, s_j) = \begin{cases} s_{\max(q-i+1,j)} & \text{if } i = q \\ & \text{or } j = 1, \\ \mathbf{1} & \text{otherwise} \end{cases}$$

$$\mathcal{I}_5(s_i, s_j) = \begin{cases} \mathbf{1} & \text{if } i \le j \\ s_j & \text{otherwise} \end{cases},$$

$$\mathcal{I}_6(s_i, s_j) = \begin{cases} s_j & \text{if } i = q \\ \mathbf{1} & \text{otherwise} \end{cases}.$$

In order to compare the behavior of the aggregation function in (6), consider the following example of Yager [12].

Suppose we have six criteria, and the standard seven-point evaluation scale (1). The evaluation of alternative a_i by expert e_k on each criterion is given in the following table:

Criteria:	c_1	c_2	c_3	c_4	c_5	c_6
Importance:	P	VH	VH	M	L	L
Score:	H	M	L	P	VH	P

Denote α_j the aggregated value of these data when we use the implication $\mathcal{I}_j$ in (6) ($j = 1, \ldots, 6$). After a routine calculation, we obtain the following results:

$$\alpha_1 = \alpha_2 = \alpha_5 = L,$$
$$\alpha_3 = M,$$
$$\alpha_4 = \alpha_6 = H.$$

5 Synthesizing Experts' Opinion

After finishing the process of the previous section, we have a vector of evaluations

$$(P_{i1}, P_{i2}, \ldots, P_{ir}) \in S^r \tag{7}$$

for each alternative $a_i \in A$, $i \in K_n$. In order to get an overall evaluation P_i for alternative $a_i \in A$ ($i \in K_n$), at the beginning we follow Yager [12]. That is, we define a function

$$Q : \{0, 1, \ldots, r\} \to S$$

so that the value $Q(i) \in S$ indicates the degree to which we want to select an alternative when a set of i experts is satisfied with that alternative.

Although the definition of Q is completely subjective, there are some rationality conditions that Q should satisfy (see Yager [12]):

1. If no experts are satisfied then the degree should be $\mathbf{0}$ (the lowest in S): $Q(0) = \mathbf{0}$.

2. If all experts are satisfied the degree should be $\mathbf{1}$ (the highest in S): $Q(r) = \mathbf{1}$.

3. If more experts agree, the degree should increase: $i > j$ implies $Q(i) \succeq Q(j)$.

In order to define a function Q which " ...can be said to emulate the usual arithmetic averaging function ...(Yager [12])", we depart from Yager's proposal, and consider a monotone nondecreasing function φ from the closed interval $[0, r]$ to the closed interval $[1, q]$ such that $\varphi(0) = 1$, $\varphi(r) = q$. If $\mathrm{Int}[a]$ denotes the integer value that is the closest to the real number a, we can define a mapping $b : \{0, 1, \ldots, r\} \to \{1, \ldots, q\}$ by

$$b(i) = \mathrm{Int}[\varphi(i)] \quad (i \in \{0, \ldots, r\}),$$

and finally Q by

$$Q_\varphi(i) = s_{b(i)}, \quad i \in \{0, 1, \ldots, r\}.$$

Note that the number of different mappings b with $b(0) = 1$ and $b(r) = q$ is equal to $\binom{r+q-2}{r-1}$. One of those was defined by Yager [12] as follows:

$$b_{(i)} = \mathrm{Int}\left[1 + i\frac{q-1}{r}\right].$$

Denote the corresponding function Q by Q_Y.

The next step is the reordering of the components of the evaluation vector (7) in descending order. Denote B_{ij} the jth highest score among $(P_{i1}, \ldots, P_{ir})$: $B_{i1} \succeq B_{i2} \succeq \ldots \succeq B_{ir}$. To find the overall evaluation, Yager [12] proposed to use the following formula:

$$P_i = \bigvee_{j=1}^{r} \left[Q_Y(j) \wedge B_{ij}\right]. \tag{8}$$

A careful study of this formula reveals that we can use the aggregation $\mathcal{M}_\vee$ introduced by Fodor and Roubens [4], with any implication and the standard negation on S. That is, in general we can define the overall evaluation of alternative a_i $(i \in K_n)$ by

$$P_i = \bigvee_{j=1}^{r} \left[\mathcal{N}(\mathcal{I}(Q_\varphi(j), \mathcal{N}(B_{ij})))\right]. \tag{9}$$

The interested reader can compare the results of the six different implications by using the same set of data.

6 Concluding Remarks

We have described some important elements of a decision process when evaluation of alternatives is based on a non-numeric linguistic scale. Such scales (equipped especially with smooth De Morgan triplets) have also been introduced into preference models (see De Baets and Fodor [1] for further details). One of our main goal was to show a class of appropriate aggregation functions like (6) and (9). On the basis of the final ratings P_i of alternatives, the decision maker can make a selection, taking into account some other (possibly subjective) criteria as well.

References

1. B. De Baets and J. Fodor, On discrete preference structures, in: B. Bouchon-Meunier, J. Gutiérrez-Ríos, L. Magdalena and R. Yager, Eds., *Technologies for Constructing Intelligent Systems: Tasks* (Springer, 2002), pp. 193–204.
2. D. Dubois and H. Prade, Weighted minimum and maximum in fuzzy set theory, *Inform. Sci.* **39** (1986) 205–210.
3. D. Dubois, H. Prade and C. Testemale, Weighted fuzzy pattern-matching, *Fuzzy Sets and Systems* **28** (1988) 313–331.
4. J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support* (Kluwer, Dordrecht, 1994).
5. J.C. Fodor, Contrapositive symmetry of fuzzy implications, *Fuzzy Sets and Systems* **69** (1995) 141–156.
6. J. Fodor, Smooth associative operations on finite chains, *Proc. 1999 EUSFLAT-ESTYLF Joint Conference* (Palma de Mallorca, Spain), 1999, pp. 359–362.
7. J. Fodor, Smooth associative operations on finite ordinal scales, *IEEE Trans. on Fuzzy Systems* **8** (2000), 791–795.
8. L. Godo and C. Sierra, *A new approach to connective generation in the framework of expert systems using fuzzy logic*, in: Proc. 18th Int. Symposium on Multiple-Valued Logic (Palma de Mallorca, Spain), Computer Society Press, Washington D.C., 1988, pp. 157–162.
9. G. Mayor and J. Torrens, *On a class of operators for expert systems*, Int. J. of Intelligent Systems **8** (1993), 771–778.
10. G.A. Miller, The organization of lexical memory, in: G.A. Talland and N.C. Waugh, Eds., *The Pathology of Memory* (New York, 1969).
11. R.R. Yager, A new methodology for ordinal multiple aspect decision based on fuzzy sets, *Decision Sciences* **12** (1981) 589–600.
12. R.R. Yager, Non-numeric multi-criteria multi-person decision making, *Group Decison and Negotiation* **2** (1993) 81–93.