Stability of asymmetrically built circular sandwich plate[*]

Lajos Pomázi

Professor, Department of Machine Construction and Safety Technics, Budapest Polytechnic, H-1081 Budapest, Népszínház u. 8. Tel: (1) 333-1500, E-mail: pomazi.lajos@bgk.bmf.hu

&

Assoc. Professor, Department of Applied Mechanics, Technical University of Budapest, H-1111 Budapest, Műegyetem rkp. 5. Tel: (1) 463 1369, E-mail: pomazi@mm.bme.hu

Abstract: *The present report gives the basic formulas on the formulation of the mechanical/mathematical models of the stability of asymmetrically built and loaded circular three-layered sandwich-type plates with (constructionally) orthotropic hard and transversally isotropic soft layers. Using tensor formalism the corresponding governing equations and natural boundary conditions are derived. As example for the stability of circular sandwich plate with isotropic hard layers the basic equations and - at Navier-type boundary conditions - method of analytical solution is given.*

Keywords: *Stability, sandwich, circular*

1 Formulation of the Problem

Our foregoing research dealt with the stability of regularly multi-layered [5] and asymmetrically built and loaded three- and multi-layered rectangular sandwich plates [4], [6]. In the latter case - continuing the proceedings - to the formulation of the stability task it was supposed, that

- material of all layers are elastic and orthotropic (transversally isotropic) or in the case of constructive anisotropy the layout of the hard layers permits to use the "effective stiffness theory", i.e. by "smoothing" (energetically) the stiffness characteristics of the reinforced layers the stiffness of equivalent flat layers can be determined;

---

[*] The connecting research was supported by the National Research Foundation on the number: OTKA N^0 T016041, which support is gratefully acknowledged.

- the "hard" layers constitute elastic plates obeying the Kirchhoff-Love laws;
- in the "soft" layers the antiplane shear stresses and in the "transversally soft" layers moreover the antiplane normal stresses are also characteristics and all these are constant across the soft layer thickness and proportional to the corresponding strains.
- according to the loading of the plate it is supposed, that the hard layers are loaded with a constant normal to the boundary membrane forces only (the shearing membrane forces are equal to zero), but this forces could be different - in particular case zero - for different layers and directions.

In the following for the investigations of the stability of circular sandwich plate all these suppositions are saved.

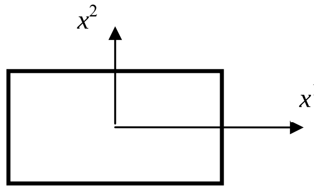
The main differences in the tasks of the stability of the rectangular and circular sandwich plates are in the geometry. Many of authors at formulation of different tasks of circular plates for deriving the corresponding strain components simply are going out from their form in the Descartes coordinates and use transformation rules between the coordinate systems [2], [3]. This method can be work well in simply tasks, but in the case of more complex structures some members of the basic equations can be missed. To avoid this problem in the given paper by formulation of the mathematical- mechanical model of the stability of circular sandwich plate from the beginning the correct tensor formalism was used, based on the following formulas:

Descartes coordinates:

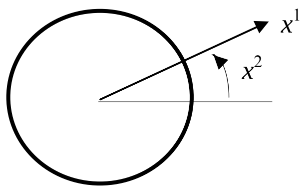
$x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$

Polar coordinates:

$x^1 \equiv r, x^2 \equiv \theta, x^3 \equiv z$



$x^\alpha(\alpha = 1,2)$



Position vector of the material points: $\underline{\mathrm{r}} = \underline{\mathrm{r}}(x^\alpha, x^3) = \underline{\bar{r}}(x^\alpha) + x^3 \underline{a}_3$

Derivatives and basic unit vectors:

$$\underline{g}_k = \frac{\partial \underline{\mathrm{r}}}{\partial x^k} = \underline{\mathrm{r}}_{,k}; \underline{a}_\alpha = \underline{\bar{r}}_{,\alpha}; \quad \underline{g}_\alpha = \underline{a}_\alpha(x^\beta) + x^3 \underline{a}_{3,\alpha}; \quad \underline{g}_3 = \frac{\underline{g}_1 \times \underline{g}_2}{|\underline{g}_1 \times \underline{g}_2|} = \underline{a}_3 \equiv \underline{k}$$

$$\underline{a}_1 = \underline{i} \cdot \cos\theta + \underline{j} \cdot \sin\theta \equiv \underline{e}_r \; ; \qquad \underline{a}_2 = -\underline{i} \cdot \sin\theta + \underline{j} \cdot \cos\theta \equiv r\underline{e}_\theta$$

Christoffel's symbols:

$$\underline{a}_{k,l} = \Gamma^m_{kl}\,\underline{a}_m \left(k,l,m = \alpha,3;\alpha = 1,2\right)$$

$$\Gamma^1_{22} = -x^1 = r \;\; ; \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{x^1} = \frac{1}{r} \;\; ; \quad \Gamma^3_{\alpha\beta} \overset{def}{=} b_{\alpha\beta} = 0 \;\; ; \quad \Gamma^\beta_{\alpha 3} = b^\beta_\alpha = 0$$

$\underline{a}_{k,l} = \underline{a}_{\alpha,\beta} + \underline{a}_{3,\beta} + \underline{a}_{3,3} = \underline{a}_{\alpha,\beta}$ **because:**

$$\underline{a}_{3,\beta} = -b^\alpha{}_\beta\,\underline{a}_\alpha = 0 \qquad \text{(Weingarten formula),}$$

$$\underline{a}_{3,3} = \Gamma^\alpha_{33}\,\underline{a}_\alpha = 0\,.$$

$$\underline{a}_{\alpha,\beta} = \Gamma^k_{\alpha\beta}\,\underline{a}_k = \Gamma^\delta_{\alpha\beta}\,\underline{a}_\delta + b_{\alpha\beta}\,\underline{a}_3 = \Gamma^\delta_{\alpha\beta}\,\underline{a}_\delta \qquad \text{(Gauss formula)}$$

$$\underline{a}_{\alpha;\beta} = \underline{a}_{\alpha\|\beta} = \underline{a}_{\alpha,\beta} - \Gamma^\delta_{\alpha\beta}\,\underline{a}_\delta = b_{\alpha\beta}\,\underline{a}_3 = 0 \qquad \text{(Weingarten's law)}$$

Covariant derivatives on the middle surfaces of the hard layers:

$$a_{\alpha\|\beta} = a_{\alpha,\beta} - \Gamma^\delta_{\alpha\beta}a_\delta, \qquad a^{\alpha\beta}\|_\gamma = a^{\alpha\beta},_\gamma + \Gamma^\alpha_{\gamma\delta}a^{\beta\delta} + \Gamma^\beta_{\delta\gamma}a^{\delta\alpha},$$

$$a^{\alpha\beta}_{\gamma}\,{}_{\|\varepsilon} = a^{\alpha\beta}_\gamma,_\varepsilon + a^{\delta\beta}_\gamma\Gamma^\alpha_{\delta\varepsilon} + a^{\alpha\delta}_\gamma\Gamma^\beta_{\delta\varepsilon} - a^{\alpha\beta}_\gamma\Gamma^\delta_{\gamma\varepsilon}, \qquad \Gamma^1_{22} = -r, \qquad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r},$$

2 Mathematical – Mechanical Modeling

Let us investigate the stability of the asymmetrically built circular sandwich plate of radius r_0 with isotropic hard and transversally isotropic soft layers of thickness h_λ (λ=1, 2) and s correspondingly, loaded by normal distributed edge forces N^{11}_λ (λ=1, 2) (Fig. 1.).

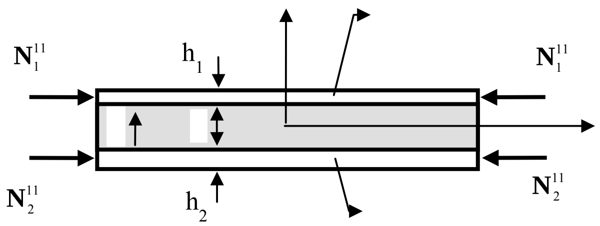


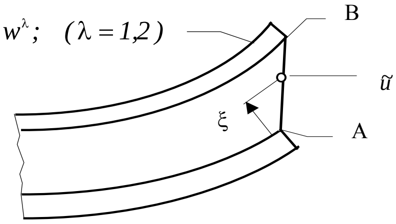
Fig. 1.

Using the common designations of the mathematical and physical quantities and the basic assumptions for the displacement – strain fields and constitutive laws we have:

2.1 Displacement fields:

For hard layers:

$$\begin{cases} \overline{u}^{\lambda}_{\alpha} \overset{def}{=} u^{\lambda}_{\alpha} + x^3\theta^{\lambda}_{\alpha} \\ \overline{u}^{\lambda}_{3} \overset{def}{=} w^{\lambda}; \quad (\lambda = 1,2) \end{cases}$$



$$\underline{u}^{\lambda} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^{\lambda} = \begin{bmatrix} u \\ rv \\ w \end{bmatrix}^{\lambda} = \begin{bmatrix} u^{\lambda} \\ rv^{\lambda} \\ w^{\lambda} \end{bmatrix}$$

$$u_k = H_k u_{x^k}$$

Lame's coefficients *(H_1=H_3=1, H_2=r)*

phisical coordinates

vector coordinates

For soft layer:

$$\underline{\tilde{u}} = \begin{bmatrix} \tilde{u} \\ r\tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} u_A \\ rv_A \\ w_A \end{bmatrix} + \frac{\xi}{s} \begin{bmatrix} u_B - u_A \\ r(v_B - v_A) \\ w_B - w_A \end{bmatrix}$$

$$u_A = u^2 - \frac{h_2}{2} w^2{}_{,1}; \quad u_B = u^1 + \frac{h_1}{2} w^1{}_{,1}$$

$$v_A = v^2 - \frac{h_2}{2} w^2{}_{,2}; \quad v_B = v^1 + \frac{h_1}{2} w^1{}_{,2}$$

$$w_A = w^2; \quad w_B = w^1$$

2.2 Strain fields:

For the hard layers - because of that’s plane-shell-type:

$$\overline{\varepsilon}_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}); \qquad \kappa_{\alpha\beta} = \theta_{\alpha|\beta},$$

and by the Kirchhoff - Love law: $\varepsilon_{\alpha 3} = 0$,

so the rotation of the layer’s normal $\theta_{\alpha} = -w_{,\alpha} - b_{\alpha}{}^{\beta} u_{\delta} = -w_{,\alpha}$

and:

$$\varepsilon_{11} = \varepsilon_r, \qquad \varepsilon_r = u_{,r},$$

$$\varepsilon_{12} = r\varepsilon_{r\theta}, \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{u_{,\theta}}{r} + v_{,r} - \frac{v}{r}\right),$$

$$\varepsilon_{22} = r^2\varepsilon_{\theta\theta}, \qquad \varepsilon_{\theta\theta} = \frac{v_{,\theta}}{r} + \frac{u}{r},$$

$$\kappa_{11} = \kappa_r , \quad \kappa_{12} = r\kappa_{r\theta} , \quad \kappa_{22} = r^2\kappa_{\theta\theta} ,$$

$$\kappa_{\theta\theta} = -w_{,rr} , \quad \kappa_{r\theta} = -\left(\frac{w_{,\theta}}{r}\right)_{,r} , \quad \kappa_{\theta\theta} = -\left(\frac{w_{,\theta\theta}}{r^2} + \frac{w_{,r}}{r}\right).$$

For the soft layers:

$$2\,\tilde{\varepsilon}_{13} = \tilde{\gamma}_{13} = \tilde{\gamma}_{rz} = \frac{1}{s}\left[u^1 - u^2 + (c_1 w^1 + c_2 w^2)_{,r}\right],$$

$$2\,\tilde{\varepsilon}_{23} = \tilde{\gamma}_{23} = r\,\tilde{\gamma}_{\theta z} = \frac{r}{s}\left[v^1 - v^2 + \frac{1}{r}(c_1 w^1 + c^2 w^2)_{,\theta}\right],$$

$$\tilde{\varepsilon}_{33} = \tilde{\varepsilon}_z = \frac{1}{s}(w^1 - w^2).$$

2.3 Constitutive laws:

For hard layers: $\underline{\varepsilon} = \underline{\underline{A}} \cdot \underline{\sigma}$, where:

$$a_{kk} = \frac{1}{E_k}, \qquad a_{ik} = -\frac{\nu_{ik}}{E_k}, \qquad (i,k,=1,2,3, i \neq k),$$

$$a_{44} = \frac{1}{G_{23}}, \qquad a_{55} = \frac{1}{G_{13}}, \qquad a_{66} = \frac{1}{G_{12}},$$

and: $\nu_{ik}E_k = \nu_{ki}E_i$, $(a_{ik} = a_{ki})$, or:

$\underline{\sigma} = \underline{\underline{B}} \cdot \underline{\varepsilon}$, where: $\underline{\underline{B}} = \underline{\underline{A}}^{-1} \longrightarrow b_{11} = \dfrac{E_1}{1-\nu_{12}\nu_{21}}, \ldots\ldots, b_{66} = G_{12}.$

For the soft layer:

$$\tilde{\tau}_{rz} = G\,\tilde{\gamma}_{rz}, \qquad \tilde{\tau}_{\theta z} = G\,\tilde{\gamma}_{\theta z}, \qquad \tilde{\sigma}_z = b_{33}\tilde{\varepsilon}_z,$$

$$E_3 = E', \quad E_1 = E_2 = E, \quad G_{13} = G_{23} = G,$$

$$\nu_{12} = \nu_{21} = \nu, \quad \nu_{13} = \nu_{23} = \nu_0, \quad \nu_{31} = \nu_{32} = \nu',$$

$$b_{33} = E'\frac{1-\nu}{1-\nu-2\nu_0\nu'}, \quad b_{44} = b_{55} = G.$$

2.4 Generalized constitutive laws:

For hard layers: $\varepsilon^*_{\alpha\beta} = \varepsilon_{\alpha\beta} + x^3\kappa_{\alpha\beta}$,

$$\underline{\varepsilon}^* = \underline{\varepsilon} + x^3\underline{\kappa} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} + x^3 \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{bmatrix}, \quad \int \underline{\sigma} dx^3 = \underline{N} = \begin{bmatrix} N^{11} \\ N^{22} \\ N^{12} \end{bmatrix}, \quad \int \underline{\sigma} x^3 dx^3 = \underline{M} = \begin{bmatrix} M^{11} \\ M^{22} \\ M^{12} \end{bmatrix},$$

$$\begin{bmatrix} \underline{N} \\ \underline{M} \end{bmatrix} = \begin{bmatrix} \underline{\underline{C}} & \underline{\underline{K}} \\ \underline{\underline{K}} & \underline{\underline{D}} \end{bmatrix} \cdot \begin{bmatrix} \underline{\varepsilon} \\ \underline{\kappa} \end{bmatrix},$$

where $\underline{\underline{C}}, \underline{\underline{K}}, \underline{\underline{D}}$ are stiffness matrices of the hard layers. Matrix $\underline{\underline{K}}$ express the coupling effect between the stretching and bending of the hard layers, which effect is usual for the (constructionally) orthotropic layer [4],[6].

For the soft layer:

$$\tilde{G} = sB = \frac{s}{\int_0^s \frac{1}{G} d\xi}, \qquad \tilde{E}_3 = sR = \frac{s}{\int_0^s \frac{1}{b_{33}} d\xi},$$

$$\tilde{C}_{13} = \tilde{C}_{23} = s^2 B, \qquad \tilde{C}_{33} = s^2 R,$$

$$\underline{\tilde{N}} = \begin{bmatrix} \tilde{T}^{13} \\ \tilde{T}^{23} \\ \tilde{N}^{33} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{13} & 0 & 0 \\ 0 & \tilde{C}_{23} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix} \cdot \begin{bmatrix} \gamma_{13} \\ \gamma_{23} \\ \gamma_{33} \end{bmatrix},$$

Relations between the physical and tensor coordinates:

$$N_r = N^{11}, \quad N_{r\theta} = rN^{12}, \quad N_\theta = r^2 N^{22}.$$

$$M_r = M^{11}, \quad M_{r\theta} = rM^{12}, \quad M_\theta = r^2 M^{22}$$

$$\tilde{T}_{rz} = \tilde{T}^{13}, \quad \tilde{T}_{\theta z} = r\tilde{T}^{23}, \quad \tilde{N}_z = \tilde{N}^{33}.$$

3 Formulation of the Boundary-value Problem

Following the method given in [1],[4] the governing equations and natural boundary conditions of the given task in the frame of our suppositions can be derived by using the Trefftz variational principle for the functional of total potential energy of the sandwich plate. The corresponding formulas are as follows:

3.1 Energy relations:

The strain energy densities for the hard and soft layers (missing the label of the hard layers: λ):

$$dU = \frac{1}{2}(\sigma^{11}\varepsilon_{11} + \sigma^{22}\varepsilon_{22} + \sigma^{12}\varepsilon_{12}), \quad d\widetilde{U} = \frac{1}{2}(\widetilde{\tau}^{13}\widetilde{\gamma}_{13} + \widetilde{\tau}^{23}\widetilde{\gamma}_{23} + \widetilde{\tau}^{33}\widetilde{\gamma}_{33}),$$

after integration of which for all the volume of the layers and calculating also the work of external forces $\hat{N}^{\alpha\beta}$ by the formulas:

$$\text{Э} = \int_{-\frac{h}{2}}^{\frac{h}{2}} dU dx^3, \quad \widetilde{\text{Э}} = \int_0^s d\widetilde{U} d\,\mathrm{x}, \quad U = \iint_{(A)} \text{Э} dA, \quad \widetilde{U} = \iint_{(A)} \widetilde{\text{Э}} dA, \quad L = \iint_{(A)} \text{Э}_L\, dA,$$

$$\text{Э} = \frac{1}{2}\left(N^{11}\varepsilon_{11} + 2N^{12}\varepsilon_{12} + N^{22}\varepsilon_{22} + M^{11}\kappa_{11} + 2M^{12}\kappa_{12} + M^{22}\kappa_{22}\right) =$$

$$= \frac{1}{2}\left(\underline{\varepsilon}\underline{\underline{C}}\underline{\varepsilon} + 2\underline{\varepsilon}\underline{\underline{K}}\underline{\kappa} + \underline{\kappa}\underline{\underline{D}}\underline{\kappa}\right),$$

$$\widetilde{\text{Э}} = \frac{1}{2}\left(\widetilde{T}^{13}\widetilde{\gamma}_{13} + \widetilde{T}^{23}\widetilde{\gamma}_{23} + \widetilde{N}^{33}\widetilde{\gamma}_{33}\right) = \frac{1}{2}\left[\widetilde{G}\left(\widetilde{\gamma}_{13}{}^2 + \widetilde{\gamma}_{23}{}^2\right) + \widetilde{E}_3\widetilde{\gamma}_{33}{}^2\right],$$

$$\text{Э}_L = \frac{1}{2}\left[\hat{N}^{11}\cdot(u_{3;1})^2 + 2\hat{N}^{12}\cdot(u_{3;1})\cdot(u_{3;2}) + \hat{N}^{22}\cdot(u_{3;2})^2\right]$$

we get the total potential energy of the plate as the functional to be minimized.

$$I = \sum_{\lambda=1}^{\lambda=2}\left(U_\lambda - L_\lambda\right) + \widetilde{U}\ .$$

3.2 Variational principle:

The Trefftz variational principle: $\delta\left(\delta_*^2 U_0\right) = 0$, where $\delta_*^2 U_0 = I$ (i.e. the second special variation of the total energy for the neutral equilibrium state is equal to the total potential energy for the real small displacements at bifurcation) was analyzed and used for derivation of basic equations of stability and vibrations of regularly multilayered plates by BOLOTIN [1] and also by the author [4,5,6]. In our case:

$$\delta I\langle u_\alpha, w\rangle = \sum_{\lambda=1}^{2}\left(\delta U_\lambda - \delta L_\lambda\right) + \delta\widetilde{U} = 0, \ \text{where:}\ \delta U_\lambda = \iint_{(A)}\left[N^{\alpha\beta}\delta u_{\alpha\|\beta} - M^{\alpha\beta}\delta w_{\|\alpha\|\beta}\right]_\lambda,$$

$$\delta\widetilde{U} = \iint_{(A)}\left[-\frac{1}{s}(-1)^\lambda\left(\widetilde{T}^{13}\delta u^\lambda + r\widetilde{T}^{23}\delta v^\lambda + \widetilde{N}^{23}\delta w^\lambda\right) + \frac{c_\lambda}{s}\left(\widetilde{T}^{13}\delta w^\lambda{}_{\|1} + \widetilde{T}^{23}\delta w^\lambda{}_{\|2}\right)\right] dA,$$

$$\delta L_\lambda = \iint_{(A)} N^{\alpha\beta} w^\lambda{}_{\|\beta}\cdot\delta w^\lambda{}_{\|\alpha}\, dA, \qquad (\alpha, \beta = 1, 2)\ .$$

Using the Gauss - Ostrogradsky theorem: $\iint_{(A)} \underline{\underline{C}} \otimes \nabla dA = \oint_L \underline{\underline{C}} \cdot \underline{n} ds$ for transformation of members having derivatives of displacement variations for the 1st variation of the functional $\mathrm{I}\langle u_\alpha, w \rangle$ we get:

$$\delta \mathrm{I}\langle u_\alpha, w\rangle = \oint_L \Big[N^{\alpha\beta} n_\beta \delta u_\alpha - M^{\alpha\beta} n_\beta \delta w_{\|\alpha} + \Big(M^{\alpha\beta}{}_{\|\beta} \cdot n_\alpha - N^{\alpha\beta} w_{\|\beta} \cdot n_\alpha +$$

$$+ \frac{c_\lambda}{s} \widetilde{T}^{\alpha 3} n_\alpha \Big) \delta w \Big] - \iint_{(A)} \left\{ \left[N^{\alpha\beta}{}_{\|\beta} + \frac{1}{s}(-1)^\lambda \widetilde{T}^{\alpha 3} \right] \delta u_\alpha + \Big[\big(M^{\alpha\beta}{}_{\|\beta} \big)_{\|\alpha} + \right.$$

$$\left. + \Big(N^{\alpha\beta} w_{\|\beta} \Big)_{\|\alpha} + \frac{c^\lambda}{s} \widetilde{T}^{\alpha 3}{}_{\|\alpha} + \frac{1}{s}(-1)^\lambda \widetilde{N}^{33} \Big] \delta w \right\} dA = 0 \,.$$

Using relations between the physical and tensor coordinates from section **2.4** we get the basic equations in physical coordinates.

3.3 Governing equations: (For the λ-th layer)

1. $$N_{r,r} + \frac{1}{r} N_{r\theta,\theta} + \frac{1}{r}\big(N_r - N_\theta \big) + (-1)^\lambda \frac{1}{s} \widetilde{T}_r = 0,$$

2. $$N_{r\theta,r} + \frac{2}{r} N_{r\theta} + \frac{1}{r} N_{\theta,\theta} + (-1)^\lambda \frac{1}{s} \widetilde{T}_\theta = 0,$$

3. $$M_{r,rr} + \frac{2}{r^2} M_{r\theta,\theta} + \frac{2}{r} M_{r\theta,r\theta} - \frac{1}{r} M_{\theta,r} + \frac{2}{r} M_{r,r} + \frac{1}{r^2} M_{\theta,\theta\theta} +$$

$$+ \frac{c^\lambda}{s}\left(\widetilde{T}_{rz,r} + \frac{1}{r} \widetilde{T}_{\theta z,\theta} \right) + (-1)^\lambda \frac{1}{s} \widetilde{N}_z - \widehat{q} = 0,$$

where $$\widehat{q} = \left(\widehat{N}_r w_{\|r} + \frac{\widehat{N}_{r\theta}}{r} w_{\|\theta} \right)_{\|r} + \left(\frac{\widehat{N}_{r\theta}}{r} w_{\|r} + \frac{\widehat{N}_\theta}{r^2} w_{\|\theta} \right)_{\|\theta} .$$

3.4 Natural boundary conditions:

$$\oint_L \left\{ \left[N_r n_r + \frac{N_{r\theta}}{r} n_\theta \right] \delta u + \left(\frac{N_{r\theta}}{r} n_r + \frac{N_\theta}{r^2} n_\theta \right) r \delta v - \left[M_r n_r + \frac{M_{r\theta}}{r} n_\theta \right] \delta w_{,r} - \right.$$

$$- \left[\frac{M_{\theta r}}{r} n_r + \frac{M_\theta}{r^2} n_\theta \right] \delta w_{,\theta} + \left[\left(M_{r,r} + \frac{M_{r\theta,\theta}}{r} + \frac{1}{r}(M_r - M_\theta) + \frac{c^\lambda}{s} \widetilde{T}_{rz} - \right.\right.$$

$$\left. - \widehat{N}_r w_{,r} - \frac{\widehat{N}_{r\theta}}{r} w_{,\theta} \right) n_r + \left(\frac{1}{r} M_{\theta r,r} + \frac{2}{r} M_{r\theta} + \frac{1}{r^2} M_{\theta,\theta} + \frac{c^\lambda}{s} \frac{1}{r} \widetilde{T}_{\theta z} - \right.$$

$$\left.\left.\left. - \frac{1}{r} \widehat{N}_{\theta r} w_{,r} - \frac{1}{r^2} \widehat{N}_\theta w_{,\theta} \right) n_\theta \right] \delta w \right\} ds = 0.$$

Natural B.C. at r_0 = const.:

1. $N_r = 0,$ **2.** $N_{\theta r} = 0,$ **3.** $M_r = 0,$

4. $M_{r,r} + \frac{1}{r}\left(M_{r\theta,\theta} + M_r - M_\theta\right) + \frac{c_\lambda}{s}\widetilde{T}_{rz} - \widehat{N}_r w_{,r} - \frac{1}{r}\widehat{N}_{r\theta} w_{,\theta} = 0.$

3.5 The boundary value problem using tangential displacements potential functions:

In the case of the boundary value problems for the rectangular sandwich-type plate it was shown [ ], that the governing equations can be simplified by using potential functions for the tangential displacements of the hard layers. Following this method – by quiet complicated calculations – it was proved, that taking potential functions $\varphi = \varphi(r,\theta)$, $\psi = \psi(r,\theta)$ for the tangential displacements $u = u(r,\theta)$, $v = v(r,\theta)$ in the form (labels: λ of the hard layers – where to write them is not necessary – are missed)**:**

$$u = \varphi_{,r} + \frac{1}{r}\psi_{,\theta}, \quad v = \frac{1}{r}\varphi_{,q} - \psi_{,r}.$$

the first two equations of section **3.3** can be written as:

$$\frac{\partial}{\partial r}[1.] + \frac{\partial}{r\partial\theta}[2.] = 0, \qquad \frac{\partial}{r\partial\theta}[1.] + \frac{\partial}{\partial r}[2.] = 0,$$

and therefore: [1.] = 0 and [2.] = 0 should be from which we get the first two basic equations written in potential functions. Introducing these functions also into the 3.-rd equations of section **3.3** finally we get the boundary value problem with these functions in form:

3.5.1 Governing equations:

[1.] $C_\lambda \Delta\varphi_\lambda + (-1)^\lambda B[\varphi_1 - \varphi_2 + (c_1 w_1 + c_2 w_2)] = 0,$

[2.] $\frac{1-\nu_\lambda}{2} C_\lambda \Delta\psi_\lambda + (-1)^\lambda B[\psi_1 - \psi_2] = 0,$

[3.] $D_\lambda \Delta\Delta w_\lambda - (-1)^\lambda R(w_1 - w_2) + N_r^\lambda \Delta w_\lambda - c_\lambda B\Delta[\varphi_1 - \varphi_2 + (c_1 w_1 - c_2 w_2)] = 0,$

where: $c_\lambda = \frac{1}{2}\left(h_\lambda + s\right)$ and $\Delta() = ()_{,rr} + \frac{1}{r}()_{,r} + \frac{1}{r^2}()_{,\theta\theta}$ is the Laplacian.

Advantage of these equations is the separation of them for w, φ and ψ .

3.5.2 Boundary conditions - simply supported edges:

$\boldsymbol{r} = \boldsymbol{r_0}$: $w^\lambda = 0,$ $M_r^\lambda = 0,$ $N_r^\lambda = 0,$ $v^\lambda = 0,$ $(\lambda = 1,2).$

Writing out these conditions (missing the label: λ) after some transformations we have:

1. $(w = 0)$: $w = 0,$

2. $(M_r = 0)$: $\Delta w - \frac{1-\nu}{r^2}(rw_{,r} + w_{,\theta\theta}) = 0.$

3. $(N_r = 0)$: $\Delta\varphi - \frac{1-\nu}{r^2}(r\varphi_{,r} + \psi_{,\theta}) = 0,$

4. $(v = 0)$: $\frac{1}{r}\varphi_{,\theta} - \psi_{,r} = 0,$

4 Sample solution for regularly built plate

For the control of basic equations derived above let us see the stability of a classical circular sandwich plate with equal thickness of isotropic hard layers ($h_\alpha = h$ = const.) and transversally isotropic core-layer of thickness: *s*. The material characteristics are as usual, and given below in the formulas:

4.1 Material parameters:

$$C_{11} = C_{22} = C = \frac{Eh}{1-\nu^2}, \quad C_{12} = \nu C, \quad C_{66} = \frac{1-\nu}{2}C,$$

$$D_{11} = D_{22} = D = \frac{Eh^3}{12(1-\nu^2)}, \quad D_{12} = \nu D, \quad D_{66} = \frac{1-\nu}{2}D.$$

4.2 Instability conditions:

In the given case atisymmetric and symmetric forms of instability can be separated by the following conditions and the corresponding equations are:

4.2.1 Antisymmetric: $w_1 = w_2, \quad \varphi_1 = -\varphi_2, \quad \psi_1 = -\psi_2$

1. $C\Delta\varphi_1 - 2B(\varphi_1 + cw_1) = 0,$

2. $D\Delta\Delta w_1 - 2Bc\Delta(\varphi_1 + cw_1) + N_r \Delta w_1 = 0,$

3. $\frac{1}{2}(1-\nu)C\Delta\psi_1 - 2\mathrm{B}\psi_1 = 0.$

4.2.2 Symmetric: $w_1 = -w_2, \quad \varphi_1 = \varphi_2, \quad \psi_1 = \psi_2$

1. $\Delta\varphi_1 = 0,$

2. $D\Delta\Delta w_1 + 2Rw_1 + N_r \Delta w_1 = 0,$

3. $\Delta\psi_1 = 0.$

4.3 Solution at Antisymmetric conditions:

Taking the **solution functions** in forms:

$$\varphi_1 = \Phi Z(r,\theta), \quad \psi_1 = \Psi Z(r,\theta), \quad w_1 = W Z(r,\theta) \quad \text{where:} \quad \Delta Z = \lambda Z(r,\theta),$$

from the set of equations of section **4.2.1** we get the **characteristic equation** of the problem:

$$g(N_r) = \lambda(\lambda - b_0)(\lambda^2 + 2a_N\lambda - c_N) = 0,$$

where parameters a_N, b_0, c_N depend from the system parameters and the load N_r:

$$a_N = \frac{D}{2}\left(N_r - \frac{2BD}{C} - 2Bc^2\right), \quad b_0 = \frac{4B}{(1-\nu)C}, \quad c_N = \frac{2BN_r}{CD}.$$

Roots of this equation are:

$$\lambda_1 = 0, \quad \lambda_2 = b_0 > 0, \quad \lambda_3 = -a_N + b_N > 0, \quad \lambda_4 = -a_N - b_N < 0,$$

where: $$b_N = \left(a_N^2 + c_N\right)^{\frac{1}{2}}.$$

Using these roots at the determination of the parameters Φ, Ψ, W in general solution functions we found, that:

$$\Phi_k = 2Bc, \qquad \Psi_k = 0, \qquad W_k = C1_k - 2B, \qquad \textbf{if} \quad (k = 1,3,4),$$

$$\Phi_2 = 0, \qquad \Psi_2 = 1, \qquad W_2 = 0, \qquad \textbf{if} \quad (k = 2)$$

Taking $\lambda_1 = k_1^2, \qquad \lambda_2 = k_2^2, \qquad \lambda_3 = k_3^2, \qquad \lambda_4 = -k_4^2$ and

using the corresponding solutions of equation $\Delta Z - \lambda Z(r,\theta) = 0$, the **general solution** we get in form:

$$\varphi_1 = 2Bc[A_1 r^n + A_3 I_n(k_3 r) + A_4 J_n(k_4 r)] \cdot sin(n\theta), \qquad \psi_1 = A_2 I_n(k_2 r) \cdot cos(n\theta),$$

$$w_1 = [-A_1 2Br^n + A_3(Ck_3^2 - 2B)I_n(k_3 r) + A_4(-Ck_4^2 - 2B)J_n(k_4 r)] \cdot sin(n\theta).$$

where $I_n(kr), \quad J_n(kr)$ **are the Bessel functions** of 1st and 2nd kind.

4.3.1 Algorithm for the determination of the critical force: N_r^*

Putting general solutions (φ_1, ψ_1, w_1) into B.C. of section **3.5.2**, we get a set of linear homogenous equations for the determination of constants A_k: $\underline{\underline{S}} \cdot \underline{A} = 0$. Condition of nontrivial solution of this equation is: $det[\underline{\underline{S}}(N_r)] = 0$.

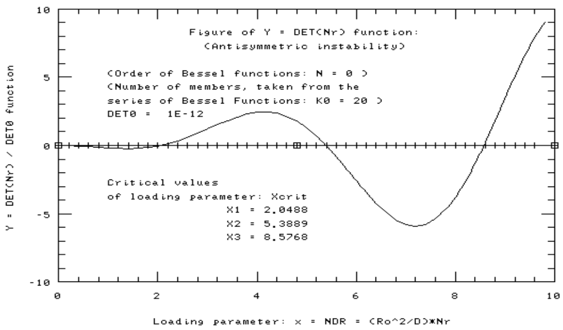
Simultaneous solution of this equation with the characteristic one gives eigen-values of the problem, minimum of which is the critical value of loading: N_r^*

4.3.2 Numerical investigations

On the basis of computer code, written for the given algorithm, the shortly shown results have been got. At control calculation (basically for one of hard layers) for

Basic Data:
Hard faces:
E = 200000 [MPa],
ν = 0.3, r_o = 0.5 [m],
h =0.002 [m],
Transversally soft core:
G = 10 [MPa], s = 0.01 [m]

(At control calculation: G=E/1E+10 → G = 0)



Min. root = Critical loading parameter these results gives: X1 = NDR = 2.048914, from which the critical value of radial UD force: min N_r = N_r^* = 2.460394E-03 [N/m], as in the literature.

Conclusions

Using tensor formalism the corresponding governing equations and natural boundary conditions are derived. As example for the stability of circular sandwich plate with isotropic hard layers the basic equations and - at Navier-type boundary conditions - method of analytical solution is given.

References

[1] Bolotin, V. V.: "*Strength, Stability and Vibration of Multilayered Plates*", Strength's Calculations, Vol.11. Moscow, 1965. pp.31 - 63 (in Russian)

[2] Kao, J.-S.: "*Bending of Circular Sandwich Plates due to Asymmetric Temperature Distribution*", AIAA Journal, Vol. 8, N° 5, May 1970, pp. 951- 954

[3] Ackermann, G.: "*Elastische, isotrope dreischichtige Kreisplatten*", Bauplanung – Bautechnik, 41. Jg., Heft 10, Oktober 1987, pp. 458 – 465.

[4] Pomázi, L.: "*Stability of Rectangular Sandwich Plates with Constructionally Orthotropic Hard Layers, Part I..",* Periodica Polytechnica, Mech. Engng, Vol.34, N° 1-2. Budapest, 1980, pp. 203-221.

[5] Pomázi, L.: "On the Stability and Post-buckling Behaviour of Multi-layer Sandwich-type Plates", Int. J. of Solids and Structures, Vol.29, N-14-15. 1992., pp. 1969-1980.

[6] Pomázi, L.: "*Stability of Asymmetrically Built and Loaded Multi-layered Rectangular Sandwich-type Plates, (Analytical solutions)*", Publications of the University of Miskolc, Series C., Mechanical Engineering, Vol.45, Miskolc,1995, pp.275-286.