Nonlinear partial differential equations treated by the generated pseudo-analysis

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Abstract: In the framework of the operations with generators extensions are considered for non-commutative and non-associative cases. The obtained results are applied through the pseudo-linear superposition principle to some nonlinear partial differential equations.

Keywords:Pseudo-analysis, semiring, partial differential equation, Burgers equation.

1 Introduction

Based on the semiring structure, it is developed in [14, 16, 19, 20] the so called idempotent analysis, and in a more general setting in [24, 25, 26, 29, 31, 33, 34] the so called pseudo-analysis in an analogous way as classical analysis, introducing $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. Pseudo-analysis was applied for the construction of solutions of nonlinear PDEs using the pseudo-linear superposition principle ([2, 10, 11, 15, 16, 14, 19, 20, 26, 27, 28, 29, 31]).

Generally we can notice quite different behavior between two classes of pseudo-operations: one which is based on generated operations (g-case) and other based on idempotent operations (sup and inf).

In the paper [38], we have considered extensions of operations $\oplus$ and $\odot$ for non-commutative and non-associative cases, which we call generalized pseudo-addition and generalized pseudo-multiplication from the right. This research was motivated by the application of the pseudo-superposition principle to nonlinear partial differential equations. Then we introduce the three parameters pair of pseudo-operations [39] and we prove that it can be used in the pseudo-linear superposition principle for the Burger's type equations, see [32].

2 Generalized pseudo-adition and pseudo -multiplication

We present in this section results obtained in [38, 39] related to the relaxation of properties of pseudo-addition and pseudo-multiplication and applications of the obtained results to nonlinear PDEs.

Definition 1 *We call real operations $\oplus$ and $\odot$ generalized pseudo-addition and generalized pseudo-multiplication (from the right), respectively, if they satisfy the following conditions:*

(i) $\oplus$ *and* $\odot$ *are functions from* $C^2(\mathbb{R}^2)$,

(ii) the equation $t \oplus t = z$ *for a given* z *has a unique solution,*

(iii) $\odot$ *is right distributive over* $\oplus$:

$$(D_r) \quad (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z).$$

Changing in the previous definition in (iii) that $\odot$ is left distributive over $\oplus$:

$$(D_l) \quad z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y),$$

we obtain generalized pseudo-addition and generalized pseudo-multiplication (from the left), respectively. Since all considerations are analogous we shall consider in the sequel only the case of generalized pseudo-addition and pseudo-multiplication from the right.

We give a representation theorem for generalized pseudo-addition and generalized pseudo-multiplication. For this purpose first we shall prove several lemmas. First of all for a function of two real variables $u = u(x, y)$ we shall use the following notation

$$(u)_1 = \frac{\partial u}{\partial x} = u_x, \quad (u)_2 = \frac{\partial u}{\partial y} = u_y, \quad (u)_{12} = \frac{\partial^2 u}{\partial x \partial y} = u_{xy}.$$

The following representation theorem is obtained in [38].

Theorem 2 *Two real operations $\oplus$ and $\odot$ from $C^2(\mathbb{R}^2)$ with $(x \odot y)_1 \neq 0$ for all x, y are generalized pseudo-addition and pseudo-multiplication (from the right), respectively, if and only if they are representable by*

$$u \odot z = h^{-1}(f(z)h(u) + \alpha(z)), \tag{1}$$

where f and α are some functions from $C^2(\mathbb{R})$, and the function h is given by

$$h(u) = \int_\gamma^u \left(e^{-\frac{1}{a} \int_\beta^s (t \oplus' t)_{12} \, dt} \right) ds,$$

the operation $\oplus'$ *is given by* $x \oplus' y = \varphi^{-1}(x \oplus y)$, *where* φ *is defined by* $\varphi(x) = x \oplus x$, *and for the operation* $\boxplus$ *which satisfies the functional equation*

$$(u \boxplus v)f(z) + \alpha(z) = (uf(z) + \alpha(z)) \boxplus (vf(z) + \alpha(z)).$$

we have

$$x \oplus y = h^{-1}((h(x) \boxplus h(y)), \tag{2}$$

Theorem 3 *If* $\oplus$ *and* $\odot$ *are generalized pseudo-addition and pseudo-multiplication (from the right), respectively, and additionally that*

(a) $x \odot 0 = a$ *(constant) for every* x;

(b) there exists y_0 *such that* $(x \odot y_0)_1 \neq 0$;

then we have

$$x \oplus y = h^{-1}(h(x) + h(y)), \quad x \odot y = h^{-1}(f(y)h(x)), \tag{3}$$

where (for γ, β *arbitrary non-zero constants)*

$$h(x) = (\gamma \oplus \beta)_2 \int \frac{(x \oplus \beta)_1}{(x \oplus \beta)_2}\, dx, \tag{4}$$

and f *is some function from* $C^2(\mathbb{R})$ *so that* $f(0) = 0$.

Remark 4 (i) Theorem 3 holds also for $\oplus, \odot \in C^1$, in this case $f \in C^1$.

(ii) As a consequence of Theorem 3 we obtain that $\oplus$ is also commutative and associative. In the special case for $f = h$ we obtain the g-calculus.

Definition 5 *For given generalized pseudo-addition* $\oplus$ *and generalized pseudo-multiplication* $\odot$ *(from the right) we call any function* h *from Theorem 2 in (1) and (2) and Theorem 3 in (3) and (4) a generator of operations* $\oplus$ *and* $\odot$.

Now we can give a complete characterization of generalized pseudo-addition and pseudo-multiplication

Theorem 6 *Let* $\oplus$ *and* $\odot$ *be a generalized pseudo-addition and a generalized pseudo-multiplication, respectively, with* $(x \odot y)_1 \neq 0$ *for all* x, y.

Then $\oplus$ *and* $\odot$ *can be represented by one of the following forms*

$$\text{(i)} \qquad x \oplus y = h^{-1}(h(y) + \psi(h(x) - h(y))),$$
$$x \odot y = h^{-1}(h(x) + \alpha(y));$$
$$\text{(ii)} \qquad x \oplus y = h^{-1}(h(y) + \psi(h(x) - h(y))),$$
$$x \odot y = h^{-1}(\alpha(y) - h(x)),$$

with some odd function ψ;

$$\text{(iii)} \qquad x \oplus y = h^{-1}(h(y) + a(h(x) - h(y))),$$
$$x \odot y = h^{-1}(h(x)f(y) + \alpha(y)),$$

where $\psi(t) = t \boxplus 0$, a *is a constant and* $\boxplus, \alpha, f, h$ *are the functions from Theorem 2.*

3 Applications to nonlinear PDEs

Example 7 The following nonlinear PDE of the first order

$$c_1(x,y)u_x + c_2(x,y)u_y = F(u), \tag{5}$$

where c_1, c_2 and F are given functions, was considered in [8] and it was shown for it the nonlinear superposition principle in the sense of Ames and Jones [11].

We shall show that there are operations $\oplus$ and $\odot$ from the family of operations characterized by Theorem 6 such that there is fulfilled the nonlinear superposition principle with respect to these operations. Taking

$$h(u) = \int \frac{dt}{F(t)},$$

we introduce using (i) from Theorem 6 generalized pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ (from the left) for arbitrary ε and for an arbitrary function α, and $\psi_\varepsilon(x) = G^{-1}(\varepsilon + G(x))$ where some function G is in C^1, by

$$u \oplus v = h^{-1}(h(v) + \psi_\varepsilon(h(u) - h(v))),$$

$$u \odot v = h^{-1}(h(v) + \alpha(u)),$$

respectively. Then using the consideration in [8] we easily obtain that if u_1 and u_2 are solutions of equation (5) then also for every ε

$$u_1 \oplus u_2 = h^{-1}(h(u_2) + \psi_\varepsilon(h(u_1) - h(u_2)))$$

is a solution of the equation (5).

For arbitrary but fixed real number a and a solution u of the equation (5) we have that also for every function α

$$a \odot u = h^{-1}(h(u) + \alpha(a))$$

is a solution of the equation (5). Namely, putting $a \odot u$ in (5) we obtain

$$c_1(x,y)(h^{-1}(h(u)+\alpha(a)))_x + c_2(x,y)(h^{-1}(h(u)+\alpha(a)))_y = \frac{1}{h'(h^{-1}(h(u)+\alpha(a)))},$$

where we have used that $F(a \odot u) = 1/h'(a \odot u)$, and h' is the derivative of h. Hence

$$\frac{c_1(x,y)h'(u)u_x}{h'((h^{-1}(h(u)+\alpha(a))} + \frac{c_2(x,y)h'(u)u_y}{h'((h^{-1}(h(u)+\alpha(a))} = \frac{1}{h'(h^{-1}(h(u)+\alpha(a)))}.$$

Since $h'(u) \neq 0$ reducing $h'(h^{-1}(h(u)+\alpha(a)))$ and using that $h'(u) = 1/F(u)$ we obtain

$$\frac{c_1(x,y)}{F(u)}u_x + \frac{c_2(x,y)}{F(u)}u_y = 1.$$

Summarizing, we have that if u_1 and u_2 are solutions of equation (5) and a_1 and a_2 arbitrary but fixed real numbers, then also for every ε and every α

$$(a_1 \odot u_1) \oplus (a_2 \odot u_2)$$

is a solution of the equation (5).

Taking specially $c_1(x,y) = c_2(x,y) = 1$ and $F(u) = u^2$ we obtain the equation

$$u_x + u_y = u^2.$$

Then we have $h(u) = -1/u$. In this case we have for the operations $\oplus$ and $\odot$

$$u \oplus v = \frac{1}{\frac{1}{v} - \psi\left(\frac{1}{v} - \frac{1}{u}\right)},$$

$$u \odot v = \frac{1}{\frac{1}{u} - \alpha(v)},$$

where $\psi(t) = -\ln(e^t + 1)$, the same statement as for the general case.

We remark that it cannot occur the situation of Theorem 3, because condition (a) from Theorem 3 would imply that $0 \odot u$ is a constant a, and therefore $(0 \odot u)_x = (0 \odot u)_y = 0$, which can give only a trivial solution $u = a$ if $F(a) = 0$.

We shall need the following special class of operations $\oplus$ and $\odot$.

Theorem 8 *Let ε be any positive real number. If k is a strictly monotone positive function and belongs to C^2, then operations defined by*

$$x \oplus y = k^{-1}(\varepsilon k(x) + k(y)), \quad x \odot y = k^{-1}(k(x)^{\varepsilon} k(y))$$

are generalized pseudo-addition and pseudo-multiplication (from the left), respectively, which belong to the class (i) in Theorem 6 (using the operation $\odot'$ defined by $x \odot' y = y \odot x$).

Remark 9 *Specially for $\varepsilon = 1$ we obtain a semiring which is based on a generator (g-calculus [26, 27]).*

Example 10 The following PDE of the second order

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, \ldots, x_n) u_{x_i x_j} + \sum_{i=1}^{n} b_k(x_1, \ldots, x_n) u_{x_k} + c(x_1, \ldots, x_n) u$$

$$= \frac{\lambda}{u} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, \ldots, x_n) u_{x_i} u_{x_j}, \qquad (6)$$

where a_{ij}, b_k, c are given functions and λ is a constant different from 1, was considered in [15].

We shall show that there are operations $\oplus$ and $\odot$ from the family of operations characterized by Theorem 8 such that there is fulfilled the nonlinear superposition principle with respect to these operations. Taking $k(x) = x^{1-\lambda}$ for $\lambda \neq 1$ and $x > 0$, we consider generalized pseudo-addition and pseudo-multiplication (from the left) from Theorem 8

$$u \oplus v = (\varepsilon u^{1-\lambda} + v^{1-\lambda})^{1/(1-\lambda)},$$

$$u \odot v = u^{\varepsilon} v,$$

respectively.

If u_1 and u_2 are positive solutions of equation (6) then also for every $\varepsilon > 0$

$$u_1 \oplus u_2 = k^{-1}(\varepsilon k(u_1) + k(u_2)))$$

is a positive solution of the equation (6).

For arbitrary but fixed real number $a > 0$ and a positive solution u of the equation (6) we have that

$$a \odot u = a^{\varepsilon} u$$

is a solution of the equation (6). Taking $a \odot (u_1 \oplus u_2)$ with $a = |\beta_0 \beta_2|^{1/\varepsilon}$ and $\varepsilon = \frac{|\beta_1|}{|\beta_2|}$ we obtain Levin's result [15].

We note that if u is a solution and a is an arbitrary positive real number then $u \odot a$ is a solution of the equation (6) if and only if $\varepsilon = 1$, which gives us the g-calculus.

4 Pseudo-operations with three parameters

In this section we introduce pseudo-operations defined with three parameters, and obtain their representations, [39]. This will enable us to find other solutions for the Burgers equations with comparison with the ones obtained in [38]. From now on we shall consider the following sets:

$$\mathcal{K}_0 = \{k \text{ strictly monotone positive function, belonging to } C^2(\mathbb{R}) \left(C^2([0,\infty))\right)\},$$

$$\begin{aligned}\mathcal{K} = \ & \{k \text{ strictly monotone positive function, belonging to } C^2(\mathbb{R}) \left(C^2([0,\infty))\right), \\ & \text{such that } k(u) \neq 0, \forall u\}.\end{aligned}$$

Proposition 11 *Let $\varepsilon_1, \varepsilon_2$ be two real positive numbers and $k \in \mathcal{K}_0$. If the operation $\oplus$ is defined by*

$$u \oplus v = k^{-1}(\varepsilon_1 k(u) + \varepsilon_2 k(v)), \tag{7}$$

then the equation $t \oplus t = z$ is solvable.

Now we consider another operation $\odot$ defined for $\delta > 0$ in this way:

$$u \odot v = k^{-1}(k(u)^{\delta} k(v)). \tag{8}$$

Theorem 12 *Given three positive parameters $\varepsilon_1, \varepsilon_2$, and δ, then the operations defined by (7) and (8) for $k \in \mathcal{K}$, are generalized operations in the sense of the Definition 1 and they belong to the class (i) of the Theorem 6, i.e., they admit the following representations:*

$$u \oplus v = h^{-1}\left(h(v) + \Psi(h(v) - h(u))\right), \tag{9}$$

$$u \odot v = h^{-1}(h(v) + \alpha(u)),$$

where $h(x) = \log(k(x)), \Psi(t) = \log(\varepsilon_1 e^{-t} + \varepsilon_2), \alpha(x) = \delta \log k(u(x))$.

Corollary 13 *For every pair of operations defined by (7) and (8) for $k \in \mathcal{K}$ there exists $\varepsilon' \in \mathbb{R}$ and an operation $\oplus_{\varepsilon''}$ for $\varepsilon'' > 0$ given by*

$$u \oplus_{\varepsilon''} v = k^{-1}(\varepsilon'' k(u) + k(v)),$$

such that $\Psi(t) = \varepsilon' + \psi(\varepsilon'' e^{-t} + 1)$, where Ψ is the function from (9) and ψ is the corresponding function from Theorem 6 (i) for the operation $\oplus_{\varepsilon''}$.

In [38] it has been proven that if u and v are solutions of the Burger's equation, then for $\varepsilon > 0$ and $a \in \mathbb{R}$ we have that $u \oplus v = k^{-1}(\varepsilon k(u) + k(v))$ and $a \odot v = k^{-1}(k(a)^{\varepsilon} + k(v))$ are solutions of the Burgers equation, too.

We shall extend this result to more general partial differential equation. Let us consider the Burger's type of nonlinear partial differential equation given by the following equality

$$u_t - \alpha u_{xx} = \alpha \Phi(u) u_x^2, \tag{10}$$

where Φ is a given continuous function and $\alpha \in \mathbb{R}$.

Theorem 14 *For equation (10), where Φ is a continuous function and $\alpha \in \mathbb{R}$, there exist three parameters pseudo-operations $\oplus$ and $\odot$ given by (7) and (8), respectively, with a generating function*

$$k(x) = \pm \int \exp\left(\int \Phi(x)\, dx\right)\, dx,$$

where the positive sign is taken when $\int \exp\left(\int \Phi(x)\, dx\right)\, dx$ is greater then zero and the negative sign is taken when $\int \exp(\int \Phi(x)\, dx)\, dx$ is less then zero, such that the pseudo-linear combination of solutions of equation (10) is also a solution of the equation (10).

Remark 15 A continuous function Φ can always be represented by some strictly monotone function k as $\Phi = \dfrac{k''}{k'}$.

Further on, for an arbitrary but fixed $\lambda \in \mathbb{R}$, we introduce the λ-corresponding equation for the equation (10) by

$$u_t - \alpha u_{xx} = \alpha \Phi_\lambda(u) u_x^2,$$

where $\Phi_\lambda = \frac{(k^\lambda)''}{(k^\lambda)'}$ and k is a generating function from Theorem 14.

Theorem 16 *Let $u_t - \alpha u_{xx} = \alpha \Phi_\lambda(u) u_x^2$ be the λ-corresponding equation for the equation (10). Then, every pseudo-linear combination of solutions of the λ-corresponding equation with respect to pseudo-operations $\oplus_\lambda$ and $\odot_\lambda$ given by generating function k^λ is also a solution of the λ-corresponding equation.*

Theorem 17 *For the Hamilton-Jacobi equation*

$$u_t - \frac{k'(u)}{k(u)} u_x^2 = 0,$$

where k is a generating function from Theorem 14, pseudo-linear combination of solutions such that $\oplus = \min$ for k being strictly decreasing function, $\oplus = \max$ for k being strictly increasing function and $x \odot y = k^{-1}(k^\delta(x) k(y))$, is also its solution.

Remark 18 Using the approach from [19] we have shown in [21] that in spite of the different behavior we can obtain the idempotent operations as a limit case of a family of generated operations. sup- and inf-measures can be obtained as limits of families of pseudo-additive measures with respect to generated pseudo-additions. The corresponding integrals with respect to sup- or inf-measures can be obtained as limits of families of g-integrals [21]. We remark that such limit procedures were used in [19] to obtain solutions of nonlinear partial differential equations.

Basing on the operations with generators, a theory was developed of generalized functions [37] in analogy to the Mikusiński's approach [22, 23]. This enables to obtain generalized solutions of the Burgers equation which are extensions of the previously obtained solutions [19, 27, 29, 33].

5 Conclusions

There are wide classes of PDEs for which the pseudo-linear superposition principle were proven. The general problem of the wider applications of the pseudo-analysis on nonlinear PDEs is how to determine that a PDE satisfy a superposition principle for suitable operations $\oplus$ and $\odot$. In this direction Goard and Broadbridge [8] have obtained a close connection of the nonlinear superposition principle and Lie symmetry algebras [4]. Since there are number of computer added algorithms for finding Lie symmetry algebra [41], this connection with Lie symmetry algebras have to be further developed to obtain algorithms for finding operations $\oplus$ and $\odot$ for wide classes of PDEs.

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