On Generalization of Fuzzy Connectives

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Abstract: In real applications of fuzzy logic the properties of aggregation operators can be crucial stage. The available aggregation operators do not always satisfy the important criteria: empirical fitting and compensation behavior. This paper introduces a new approach to define customized aggregations operators. First the concept of proposition based operators is introduces then as a special case the condition constrained distance-based fuzzy connectives are discussed.

Keywords: fuzzy connectives, t-norms, t-conorms, triangular norms

1 Introduction

Information aggregation plays a key role in wide spread applications of fuzzy logic. The great variety of available aggregation operators inspired Zimmermann to investigate the criteria for selecting aggregation operations []. He had selected eight important rules according to which operators can be used in a specific situation or model. From practical point of view "Empirical Fitting" and "Compensation" behavior are two important criteria.

Empirical Fitting means that the operators have to be appropriate models of the real system behavior. On the other hand in real applications, for example at decision making it would be required that a higher degree of satisfaction of one of the criteria could be compensated for a lower degree of satisfaction of another criterion to a certain extent. In this sense, union provides full compensation, while in case of intersection there is no compensation at all. An operator M is said to be a *compensative* if

$$\min(x, y) \leq M(x, y) \leq \max(x, y),\ \forall(x, y) \in [0,1]^2. \tag{1}$$

These two criteria have motivated the introduction of proposition- and condition based operators, which can satisfy the requirements of empirical fitting and compensation and also other custom defined properties.

The paper is organized as follows; firstly the basic idea of proposition based operators is introduced together with some examples, next the concept of distance-based operators with respect to constrain membership function is discussed.

2 Proposition-based Fuzzy Coonectives

Definition

Let *A, B be* two fuzzy subsets of the universe of discourse *X* and **P** is an arbitrary proposition defined on *X* with truth values $v(P) \in [0,1]$. The *proposition based operator on A and B with respect to* **P** is defined as

$$O_{\mathbf{P}}(A,B) = \{(x, \mu_{O_{PY}}(x)) | x \in X,\ \mu_{O_{\mathbf{P}}}(x) \in [0,1]\}, \tag{2}$$

$$\mu_{O_{\mathbf{P}}} : x \mapsto f(\mu_A(x), \mu_B(x), v(\mathbf{P}(x))). \tag{3}$$

The proposition **P** can be a function of the membership functions of the two fuzzy sets. In this case the definition is slightly modified as follows.

Definition

Let *A, B be* two fuzzy subsets of the universe of discourse *X* and P is a proposition defined on the two variables μ_A and μ_B with truth values $v(P) \in [0,1]$. The *proposition based operator on A and B with respect to* P is

$$O_{\mathbf{P}}(A,B) = \{(x, \mu_{O_{\mathbf{P}}}(x)) | x \in X,\ \mu_{O_{\mathbf{P}}}(x) \in [0,1]\}, \tag{4}$$

$$\mu_{O_{\mathbf{P}}} : x \mapsto f(\mu_A(x), \mu_B(x), v(\mathbf{P}(\mu_A(x), \mu_B(x)))). \tag{5}$$

Special Cases

Let the proposition be defined as follows

$$v(\mathbf{P}(x)) = \begin{cases} 1 \text{ if } \mu_A(x) \le \mu_B(x) \\ 0 \quad \mu_A(x) > \mu_B(x) \end{cases}. \tag{6}$$

If the membership function of the operator is defined by

$$\mu_{O_{\mathbf{P}}} : x \mapsto \begin{cases} \mu_A(x) \text{ if } v(\mathbf{P}) = 1 \\ \mu_B(x) \text{ if } v(\mathbf{P}) = 0 \end{cases}, \tag{7}$$

then the conventional Zadehian-minimum operator is obtained.

If $v(\mathbf{P}(x)) = constant, \forall x \in X$ and $f : x \mapsto \mu_A(x)\mu_B(x)$ then the algebraic product is generated.

As the proposition is defined on X and its values $v(P) \in [0,1]$ it can be modeled by the fuzzy subset of X, as

$$P = \{(x, \mu_P(x)) | x \in X,\ \mu_P(x) = v(\mathbf{P}(x))\}. \tag{8}$$

By means of this modeling the proposition based operator is interpreted as a *condition based operator.*

Definition

Let *A, B* and *P* be three fuzzy subsets of the universe of discourse *X*. The *condition based operator on A and B with respect to P* is

$$O_P(A,B) = \{(x, \mu_{O_P}(x)) | x \in X,\ \mu_{O_P}(x) \in [0,1]\}, \tag{9}$$

$$\mu_{O_P} : x \mapsto f(\mu_A(x), \mu_B(x), \mu_P(x)). \tag{10}$$

If the function *f* does not depend on the membership function of *P* then a binary fuzzy relation is obtained which can be in special cases either a t-norm or a t-conorm.

3 Condition constrained distance-based operators

Definition Let *A, B* and *Y* be three fuzzy subsets of the universe of discourse *X*. The *minimum distance operator on A and B with respect to Y* is defined as

$$I_{DY}(A,B) = \min_{DY}(A,B) = \{(x, \mu_{I_{DY}}(x)) | x \in X,\ \mu_{I_{DY}}(x) \in [0,1]\}, \tag{11}$$

$$\mu_{I_{DY}} : x \mapsto \begin{cases} \mu_A(x) & \text{if } |\mu_A(x) - \mu_Y(x)| \le |\mu_B(x) - \mu_Y(x)| \\ \mu_B(x) & \text{if } |\mu_B(x) - \mu_Y(x)| < |\mu_A(x) - \mu_Y(x)| \end{cases}. \tag{12}$$

Properties

It is obvious that the operator is commutative, and associative, hence from an algebraic point of view, I_{DY} is a commutative semigroup operation on [0,1].

1. If $\mu_Y \equiv 1$ then $I_{DY}(A,B) = \max(A,B)$.

Proof. If $\mu_A(x) \le \mu_B(x)$ then $|\mu_A(x) - 1| \ge |\mu_B(x) - 1|$ and $\mu_{I_{DY}} : x \mapsto \mu_B(x)$ Due to the commutativity of the operator the role of the two membership functions is symmetric, so the proposition holds. ■

2. If $\mu_Y \equiv 0$ then $I_{DY}(A,B) = \min(A,B)$.

Proof. If $\mu_A(x) \le \mu_B(x)$ then $|\mu_A(x) - 0| \le |\mu_B(x) - 0|$ and $\mu_{I_{DY}} : x \mapsto \mu_A(x)$ and because of commutativity the proposition holds. ■

3. If μ_Y is an arbirary membership function, neither identical to zero or 1 then nor 1 nor 0 are identity elements.

Proof.

Let be $\mu_A(x) = 1$, $\mu_B(x) = 0.1$ and $\mu_Y(x) = 0.9$. Then $\mu_{I_{DY}}(x) = \mu_A(x)$.

Let be $\mu_A(x) = 0$, $\mu_B(x) = 0.9$ and $\mu_Y(x) = 0.1$. Then $\mu_{I_{DY}}(x) = \mu_A(x)$. ■

4 Distance-based operators

Distance-based operators introduced by Rudas [2] are special cases of the condition constrained distance-based operators. In this case the membership function of the fuzzy set *Y* is constant. The basic definitions are recalled next.

Let *e* be an arbitrary element of the closed unit interval [0,1] and denote by $d(x, y)$ the distance of two elements *x* and *y* of [0,1]. The idea of definitions of distance-based operators is generated from the reformulation of the definition of the min and max operators as follows

$$\min(x,y) = \begin{cases} x, \text{ if } d(x,0) \le d(y,0) \\ y, \text{ if } d(x,0) > d(y,0) \end{cases}, \quad \max(x,y) = \begin{cases} x, \text{ if } d(x,0) \ge d(y,0) \\ y, \text{ if } d(x,0) < d(y,0) \end{cases}$$

Definition

The *maximum distance minimum operator with respect* to $e \in [0,1]$ is defined as

$$\max_e^{\min}(x,y) = \begin{cases} x, & \text{if } d(x,e) > d(y,e) \\ y, & \text{if } d(x,e) < d(y,e) \\ \min(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{13}$$

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5 Modified distance-based operators.

By using additional conditions the operators can be transformed into distance-based operators having neutral elements.

Definition

Let *A, B* and *Y* be three fuzzy subsets of the universe of discourse *X*. The *minimum distance minimum of A and B with respect to Y* is defined as

$I_Y^D(A,B)=\min_{DY}(A,B)=\{(x,\mu_{I_Y^D}(x))|x\in X,\ \mu_{I_Y^D}(x)\in[0,1]\}$,

$$\mu_{I_Y^D}:x\mapsto\begin{cases} \mu_A(x) & \text{if } |\mu_A(x)-\mu_Y(x)|<|\mu_B(x)-\mu_Y(x)| \\ \mu_B(x) & \text{if } |\mu_B(x)-\mu_Y(x)|>|\mu_A(x)-\mu_Y(x)| \\ \min(\mu_A(x),\mu_B(x)) & \begin{array}{l}\text{if } |\mu_A(x)-\mu_Y(x)|=|\mu_B(x)-\mu_Y(x)| \text{ and} \\ \mu_A(x)\mu_B(x)=0 \text{ or } \mu_B(x)=1 \text{ or } \mu_A(x)=1\end{array} \end{cases} \tag{17}$$

Definition

Let *A, B* and *Y* be three fuzzy subsets of the universe of discourse *X*. The *maximum distance maximum of A and B with respect to Y* is defined as

$$U_Y^D(A,B)=\max_{DY}(A,B)=\{(x,\mu_{U_Y^D}(x))|x\in X,\ \mu_{U_Y^D}(x)\in[0,1]\}, \tag{18}$$

$$\mu_{I_Y^D}:x\mapsto\begin{cases} \mu_A(x) & \text{if } |\mu_A(x)-\mu_Y(x)|>|\mu_B(x)-\mu_Y(x)| \\ \mu_B(x) & \text{if } |\mu_B(x)-\mu_Y(x)|>|\mu_A(x)-\mu_Y(x)| \\ \max(\mu_A(x),\mu_B(x)) & \text{if } |\mu_A(x)-\mu_Y(x)|=|\mu_B(x)-\mu_Y(x)| \end{cases} \tag{19}$$

Conclusions

In this paper the concept of application oriented propositions based fuzzy connectives are introduced. As a special case the general discussion of condition constrained distance-based fuzzy connectives is given.

References

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