Exponential Stabilization of Robot Arms

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Abstract: *In this paper it is considered a special oriented control method which can be used for control of systems described by Lagrange's equations in the form useful especially for control of mechanical systems. This method can be ranked among Lyapunov based methods. There is proved a theorem shoving that the control process is exponentially stable. The control schemes can be applied generally to mechatronical systems, especially for robots.*

Keywords: *Exponential Stabilization, Labrange's Equations*

I INTRODUCTION

The motion of a mechanical system is able to describe by Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \qquad (1)$$

for $j=1,\dots,n$; where Q_j are the non-conservative generalised forces, the Lagrangian is defined as $L=K-V$, where V represents the potential energy and K is the kinetic one. The kinetic energy can be written in the quadratic form

$$K = \frac{1}{2}\dot{q}^T H(q)\dot{q} \ . \qquad (2)$$

The matrix $H(q)$ is the inertial matrix including inertial terms of inertial load distributions of actuators and u is the vector of input torques generated at joint actuators. Our problem is to define any controller if possible in any simple form that will be exponential stable [1] and then to set all definable parameters. All computations can be simulated on PC.

II DESCRIPTION OF SYSTEM

The equation of motion for a studied system is possible to rewrite from (1) into the set of differential equations in a vector form

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u \ , \qquad (3)$$

where

$$C(q,\dot{q}) = \left[\frac{1}{2}\dot{H}(q) + N(q,\dot{q})\right] \qquad (4)$$

is a matrix, $q = (q_1,\dots,q_n)^T$ is a vector of generalised co-ordinates that is complete and independent, for this purpose it is usually used a matrix method; further let be denoted

$$g(q) = (\partial V/\partial q_1,\dots,\partial V/\partial q_n)^T \ ,$$

V is a generalised potential function, $H(q)$ is a symmetric matrix (positive definite and continuous), N is a skew symmetric matrix in a form

$$N(q,\dot{q}) = \frac{1}{2}\left[\sum_{k=1}^{n}\dot{q}_k \frac{\partial H_{i,k}}{\partial q_j} - \sum_{k=1}^{n}\dot{q}_k \frac{\partial H_{jk}}{\partial q_i}\right] . \qquad (5)$$

Furthermore, the form

$[\frac{1}{2}\dot{H}(q)+N(q,\dot{q})]\dot{q}$

is quadratic in $\dot{q}$.

It can be proved that this form has every cinematic chain, for example industrial robots, cranes, excavators and so on. It will be suitable to have some inequalities for the other work. But we will not use all of them.

A) It is known that for every constant symmetric positive definite matrix Λ and every vector x is

$\lambda_{\min} x^T x \le x^T \Lambda x \le \lambda_{\max} x^T x$,

where $\lambda_{\min} = \lambda_{\min}(\Lambda)$ is the smallest ($\lambda_{\max} = \lambda_{\max}(\Lambda)$ … biggest) eigenvalue of matrix Λ .

B) Matrix $H = H(q)$ is continuous and the working space is compact, therefore there are constant n x n matrices K_1, K_2 so that the inequalities $K_1 \le H(q) \le K_2$ hold. But even without compactness matrix $H(q)$ is positive definite and symmetric therefore there is strictly positive minimum λ_m (and maximum λ_M) eigenvalue of H(q) for all configurations q and for every x so that $\lambda_m x^T x \le x^T H(q) x \le \lambda_M x^T x$.

C) Matrix N is skew-symmetric and so $x^T N(q) x = 0$ for every q and x.

D) The function V is bounded and $\|g(q)\| \le K_g$ for some positive constant $K_g > 0$.

E) $\|C(q,\dot{q})\| \le K\|\dot{q}\|$ for some constant $K > 0$.

Now we suppose a target position q_F and a velocity $\dot{q}_F$ are given and consider a set-point problem where any initial state $(q(0),\dot{q}(0))$ is allowed to approach asymptotically to the defined target, or final, state $(q,\dot{q}) = (q_F,\dot{q}_F)$. Remember, here the state is represented by ordered pairs $(q,\dot{q})$ … position and velocity.

III ALGORITHM OF CONTROL

Let $\Delta q = q - q_F$ be a difference and define vectors y, z :

$$\dot{y} = \dot{q}_F - A\Delta q \quad \text{and} \quad z = \dot{q} - \dot{y} \qquad (6)$$

where A is constant and positive definite matrix. Consider the control law in the following form (compare with (3)):

$$u = H(q)\ddot{y} + C(q,\dot{q})\dot{y} + g(q) - Bz \qquad (7)$$

for control system (3), the matrix B is positive definite too. If we substitute (7) into (3), then with using (6) is

$$H(q)\dot{z} + C(q,\dot{q})z + Bz = 0 \quad , \qquad (8)$$

which describes a first-order differential equation in the new variable z. This variable z is related to the tracking error as above. From (6) we have

$$\Delta\dot{q} = z - A\Delta q \qquad (9)$$

Theorem:

The control law (7) has the following properties:

(a) $z \in L_2 \cap L_\infty$ and $\|z(t)\| \le k_1 \exp(-at)\|z(0)\|$

(b) $\Delta q \in L_2 \cap L_\infty$ and $\|\Delta q(t)\| \le k_2 \exp(-bt)(\|\Delta q(0)\| + \|\Delta\dot{q}(0)\|)$

(c) $\Delta\dot{q} \in L_2 \cap L_\infty$ and $\|\Delta\dot{q}(t)\| \le k_3 \exp(-ct)(\|\Delta q(0)\| + \|\Delta\dot{q}(0)\|)$

where k_1, k_2, k_3, a, b, c are positive constants. (See L_2 and L_∞ to [2].)

Proof:

Ad (a) Let the following positive definite function be defined

$$V = \frac{1}{2} z^T H(q) z \quad (10)$$

then its derivative with respect the time t is

$$\dot{V} = -z^T B z \le 0 \ . \quad (11)$$

The function V is decreasing and positive, therefore

$$0 \le \int_0^t z^T B z \, d\tau = -\int_0^t \dot{V}(\tau) d\tau = V(0) - V(t) \le V(0) \quad (12)$$

From property A) as above we obtain

$$z^T z \le z^T B z / \lambda_{\min}(B)$$

and so

$$\int_0^t \|z(t)\|^2 d\tau \le \frac{1}{\lambda_{\min}(B)} \int_0^t z^T B z \, d\tau \le \frac{V(0)}{\lambda_{\min}(B)} < \infty \quad (13)$$

We see that $z \in L_2$. The function V is bounded, therefore Z is bounded too and hence $z \in L_\infty$.

Now prove inequality of (a)

$$\frac{\dot{V}}{V} = -2 \frac{z^T B z}{z^T H(q) z} \le -2 \frac{\lambda_{\min}(B) z^T z}{\lambda_M \|z\|^2} = -2 \frac{\lambda_{\min}(B)}{\lambda_M} = -2a \quad (14)$$

and so integration of (14) yields

$$\ln \frac{V(t)}{V(0)} = \int_0^t \frac{\dot{V}}{V} d\tau \le -2at \quad (15)$$

and then we obtain the following inequality for V:

$$V(t) \le V(0) \exp(-2at) \ . \quad (16)$$

From properties B), (14) and (16) we obtain the inequalities

$$\lambda_m z^T z / 2 \le z^T H(q) z / 2 \le z^T(0) H(q(0)) z(0) e^{-2at} / 2 \le \|Z(0)\|^2 \lambda_M e^{-2at} / 2$$

and therefore we derive

$$\|z(t)\|^2 \le \frac{\lambda_M}{\lambda_m} \|z(0)\|^2 \exp(-2at) \ . \quad (17)$$

We see that the inequality of (a) is true for some constant k_1 .

Ad (b) We will prove the inequality of (b), because the other's follow from this inequality. The solution of (9) is

$$\Delta q(t) = \exp(-At) \Delta q(0) + \int_0^t \exp(-A(t-\tau)) z(\tau) \, d\tau \quad (18)$$

and from this for any suitable constants we can derive

$$\|\Delta q(t)\| \le \|\exp(-At)\| \cdot \|\Delta q(0)\| + \|\exp(-At)\| \cdot \int_0^t \|\exp(A\tau)\| \cdot \|z(\tau)\| d\tau \le$$
$$\le n_1 \exp(-lt) \|\Delta q(0)\| + n_2 \exp(-lt) \int_0^t \exp(l\tau) \|z(\tau)\| d\tau$$

where $l \ne a$ and if we use (a) which was proved then

$$\|\Delta q(t)\| \le n_1 \exp(-lt) \|\Delta q(0)\| + n_2 k_1 \|z(0)\| \exp(-lt) \int_0^t \exp((l-a)\tau) d\tau$$
$$= n_1 \exp(-lt) \|\Delta q(0)\| + n_2 k_1 \|z(0)\| [\exp(-at) - \exp(-lt)] / (l-a) \quad (19)$$

Define $b = \min\{l, a\}$. From (9) it follows

$$z(0) = A \Delta q(0) + \Delta \dot{q}(0)$$

and if we substitute this one into (19) we obtain

$$\|\Delta q(t)\| \le \exp(-bt)\begin{bmatrix}\alpha\|\Delta q(0)\| + \\ \beta\|\Delta\dot{q}(0)\|\end{bmatrix} \le$$

$$k_2 \exp(-bt)\left[\|\Delta q(0)\| + \|\Delta\dot{q}(0)\|\right] \qquad (20)$$

for constants defined by the following equalities

$\alpha = n_1 + 2\|A\|k_1 n_2 / |l - a|$,

$\beta = 2k_1 n_2 / |l - a|$,

$k_2 = \max\{\alpha, \beta\}$.

From the last inequality it is now very easy to obtain the inequality (b). The other parts of (b) can be easy proved from this inequality.

(c) This proof is easy to make with using (9), (a) and (b), hence we will leave it to reader. The theorem is now proved.

Conclusion

This methods was used for control of laboratory go-car, for control of mechanical system obtained by reconstruction of an old plotter and for control of robot arms. The experiences are very good. Along a transformation of (3) to the state stable follows from the general theory of control that a pertinent Jacoby matrix is stable.

References

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