Fuzzy Differential Equations in Modeling of Hydraulic Differential Servo Cylinders

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Abstract: Hydraulic differential electric servo cylinders are electromechanical tools applicable for driving e.g. manipulators. Actual models are strongly nonlinear, coupled systems of differential equations. In the present paper we propose a new model using fuzzy differential equations under strongly generalized differentiability concept. The key point is a continuous fuzzyfication of the signum function. Numerical solutions of the fuzzy differential equations are proposed by using an Euler-type method.

1 Introduction

Hydraulic drives have advantages with respect to DC motor driven systems in several applications. The actual classical models of the Hydraulic Differential Cylinders (HDC) are systems of highly nonlinear coupled differential equations. (see e.g. [4], [10s]). The most important phenomena influencing their behavior, as e.g. warming up of the sliding surfaces are determined by local effects, and cannot be modeled accurately by a crisp system. Also the actual models of the friction forces show discontinuous variation at the zero transition of the piston's velocity that is a locally nonlinearizable nonlinearity in the classical theory of dynamical

[1] The authors acknowledge the support of Project ID 17 of Hungarian-Romanian Intergovernmental Science & Technology Cooperation Program.

systems. A common PID-type controller may generate noise-like acceleration signal due to feeding back the effects of such fluctuations. Warming up of the working fluid during operation influences these friction properties, too. Dynamic interaction between the system and its environment neither measured nor modeled is another agent influencing the system's behavior.

The above discussion shows us that, the classical model of HDC contains several terms which are subject of non-statistical uncertainty. The frictional term and the exterior disturbance force are the main such terms. In order to take into account these uncertainties and to build up a more accurate model we propose in the present paper the use of fuzzy differential equations (FDEs).

Fuzzy differential equations appear natural as tools for modeling dynamical systems under uncertainty. Till now, they are rarely used in modeling real-world systems since their theory was developed relatively recently. As it is shown in several recent papers, FDEs are not just an easy extension of the theory of ODEs to the fuzzy case (see e.g. [7], [8], [1]). This fact is also slowing down the extension of the applicability of fuzzy differential equations. There are several different interpretation of the notion of FDE (for a discussion about them please refer to [1]). In the present paper we will use the so called strongly generalized differentiability introduced recently as a method which solved some problems with the other FDE concepts as H-derivative (see [9]) or fuzzy differential inclusions (see [5]).

Strongly generalized differentiability was introduced in [2]. The strongly generalized derivative is defined for a larger class of fuzzy-number-valued functions than the H-derivative and fuzzy differential equations can have solutions with decreasing length of their support (this was not the case for the H-derivative). Also, contrary to the case of differential inclusions, the derivative concept for fuzzy-number-valued function is defined and this makes this method more appropriate for numerical computation. First order linear fuzzy differential equations are investigated in [3] and the behavior of their solutions motivate us also to use the above cited results in the present paper for building a novel model for HDCs.

The key point in our discussion is how to fuzzify the classical model in order to get meaningful conclusions. The key role in this fuzzification is that of the frictional term. In the present paper, following [5], we fuzzify the Signum function, but in a different way, by making this term also continuous! However we have gained the continuity of the frictional term, since it is a fuzzy one, we obtain a fuzzy solution for our model. The interpretation of this model is the fuzzy set of all attainable trajectories of the system. In the present paper we do not deal with the problem of control, this being subject of further research.

After a preliminary section we propose in Section 2 the fuzzy Frictional term, which becomes in this way continuous in contrast with its crisp correspondent. Here we present also some preliminary results how friction is modeled by using this function. In Section 3 we propose the fuzzy model of the Hydraulic Differential Servo Cylinder. We end up with some conclusions and further research topics.

2 Preliminaries

We denote by $\mathbf{R}_F$ the space of fuzzy numbers. For $0 < r \leq 1$, denote $[u]^r = \{x \in \mathbf{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbf{R};\ u(x) > 0\}}$. Then it is well-known that for any $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For $u, v \in \mathbf{R}_F$, and $\lambda \in \mathbf{R}$, the sum $u + v$ and the product $\lambda \cdot u$ are defined by $[u+v]^r = [u]^r + [v]^r$, $[\lambda \cdot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$.

Let D : $\mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}_+ \cup \{0\}$,

$$D(u,v) = \sup_{r \in [0,1]} \max\{| u_-^r - v_-^r |, | u_+^r - v_+^r |\},$$

be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u_-^r, u_+^r]$, $[v]^r = [v_-^r, v_+^r]$. In this case $(\mathbf{R}_F, D)$ is a complete metric space. The above operations and the metric space structure allows us to build a mathematical analysis over the space of fuzzy numbers, however some problems appear due to the lack of some properties.

The so called H-difference or Hukuhara difference will play a key role in the present paper. Let us recall its definition.

Definition 1 *(see e.g. [9])* *Let $x, y \in \mathbf{R}_F$. If there exists $z \in \mathbf{R}_F$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$.*

In this paper the "$\ominus$" sign stands always for H-difference and let us remark that $x \ominus y \neq x + (-1)y$. We will denote for simplicity $x + (-1)y$ by $x - y$.

Let us recall the definition of strongly generalized differentiability proposed in [2].

Definition 2 *Let $f : (a,b) \to \mathbf{R}_F$ and $x_0 \in (a,b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbf{R}_F$, such that*

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

(ii) for all $h>0$ *sufficiently small,* $\exists f(x_0)\ominus f(x_0+h),\quad f(x_0-h)\ominus f(x_0)$ *and the limits*

$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0+h)}{(-h)}=\lim_{h\searrow 0}\frac{f(x_0-h)\ominus f(x_0)}{(-h)}=f'(x_0),$$

or

(iii) for all $h>0$ *sufficiently small,* $\exists f(x_0+h)\ominus f(x_0),\quad f(x_0-h)\ominus f(x_0)$ *and the limits*

$$\lim_{h\searrow 0}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\searrow 0}\frac{f(x_0-h)\ominus f(x_0)}{(-h)}=f'(x_0),$$

or

(iv) for all $h>0$ *sufficiently small,* $\exists f(x_0)\ominus f(x_0+h),\quad f(x_0)\ominus f(x_0-h)$ *and the limits*

$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0+h)}{(-h)}=\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0).$$

(h *and* $(-h)$ *at denominators mean* $\frac{1}{h}\cdot$ *and* $-\frac{1}{h}\cdot$*, respectively).*

We say that a function is (i)-differentiable if it is differentiable as in the previous Definition gdif1, (i), etc.

The following theorems concern the existence of solutions of a fuzzy initial value problem under generalized differentiability (see [2]).

Theorem 1 *Under some relaxed conditions (for which the reader is asked to consult [2] the fuzzy initial value problem*

$$\begin{cases} y'=f(x,y)\\ y(x_0)=y_0\end{cases}$$

has two solutions (one (i)-differentiable and the other one (ii)- differentiable) $y,\overline{y}\ :[x_0,x_0+r]\to B(y_0,q)$ *and the successive iterations*

$$y_0(x)=y_0$$

$$y_{n+1}(x)=y_0+\int_{x_0}^{x} f(t,y_n(t))dt,$$

and

$$\overline{y}_0(x)=y_0$$

$$\overline{y}_{n+1}(x) = y_0 \ominus (-1) \cdot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt$$

converge to these two solutions respectively.

According to the previous Theorem Peano, we restrict our attention to functions which are (i) or (ii)-differentiable on their domain except a finite number of points (see also [2]).

The FDEs will have in the present paper will have input data trapezoidal fuzzy numbers. We recall that for $a < b < c < d$, $a,b,c,d \in \mathbf{R}$, the trapezoidal fuzzy number $u = (a,b,c,d)$ determined by a,b,c and d is given such that $u_-^r = a + (b-a)r$ and $u_+^r = d - (d-c)r$ are the endpoints of the $r-$ level sets, for all $r \in [0,1]$.

3 The Fuzzy Friction Term

The key point in our model is played by the friction term. Since the friction term's (Stribec's force) discontinuity is induced by the signum function. According to several authors this term is responsible for the fact that the model consists of highly nonlinear differential equations.

Recently, in [5] the friction term was replaced by a fuzzy term. Then the equation of dry friction was modeled by fuzzy differential equations, in that case the interpretation being as system of differential inclusions. In our case we fuzzify the signum function similarly to [5], but simultaneously we transform it to a continuous term. As a conclusion, the signum function will be in our model continuous fuzzy-valued function and the friction force as well. The trapezoidal-valued signum function is

$$Sgn_{\varepsilon,\delta}(v) = \begin{cases} -1, \text{ if } \overline{v} \le -\varepsilon \\ (-1,-1+\delta,1-\delta,1) \ominus \left(-\frac{2}{\varepsilon}, -\frac{2+\delta}{\varepsilon}, -\frac{\delta}{\varepsilon}, 0\right) \cdot \overline{v}, \text{ if } \left|\overline{v}\right| < \varepsilon \\ 1, \text{ if } \overline{v} > \varepsilon \end{cases}. \tag{1}$$

It is easy to see that

$$\lim_{\varepsilon,\delta \to 0} Sgn_{\varepsilon,\delta}(v) = \begin{cases} -1, \text{ if } \overline{v} \le -\varepsilon \\ [-1,1], \text{ if } \left|\overline{v}\right| < \varepsilon \\ 1, \text{ if } \overline{v} > \varepsilon \end{cases},$$

which coincides with the interval-valued signum function proposed in [5] (the convergence is understood surely only pointwise).

4 Experimental Results

In this section we solve the equation of dry friction. This problem was studied also in [5], employing the interpretation with differential inclusions. Let us remark here, that in [5] the existence of periodic solutions is investigated, but the equation is not solved. In the present paper we solve numerically this equation under different types of control signals.

The fuzzy differential equation modeling dry friction is

$$y'' + \alpha y' + \mu \cdot Sgn(y') + y = u(t), \tag{2}$$

where $\alpha, \mu \in \mathbf{R}$ are positive constants, $u(t)$ is a control signal and the signum function $Sgn(y')$ is given by (1) in our model. The coordinate $y : \mathbf{R} \to \mathbf{R}_F$ is considered to be a trapezoidal fuzzy number valued function. The initial conditions are considered to be crisp values.

In order to solve the equation we rewrite it as a system of first order FDEs as follows

$$\begin{cases} y' = v \\ v' = u(t) - \alpha v - \mu \cdot Sgn(v) - y \end{cases}.$$

The initial values are $y(0) = 0$ and $v(0) = 1$ and the control signal is $u(t) = \sin t$.

Each of the above equations may have locally two solutions given by Theorem Peano

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} v_n dt, \text{ or}$$
$$\overline{y}_{n+1}(t) = y_0 \ominus (-1)\int_{t_0}^{t} v_n dt,$$

and

$$v_{n+1}(t) = v_0 + \int_{t_0}^{t} (u(t) - \alpha v_n - \mu \cdot Sgn(v_n) - y_n) dt \text{ or}$$
$$v_{n+1}(t) = v_0 \ominus (-1)\int_{t_0}^{t} (u(t) - \alpha v_n - \mu \cdot Sgn(v_n) - y_n) dt.$$

In order to solve the problem we will employ a numerical method based on the classical Euler method. We consider the approximation given by this method sufficient for our purposes. Surely theoretical study and implementation of more sophisticated methods is subject of future research.

One step of the Euler's method in our case is given by

$$y(t+h) = y(t) + hv(t), \text{ or}$$
$$y(t+h) = y(t) \ominus (-1)hv(t)$$

and

$$v(t+h) = v(t) + h(u(t) - \alpha v(t) - \mu \cdot Sgn(v(t)) - y(t)) \text{ or}$$
$$v(t+h) = v(t) \ominus (-1)h(u(t) - \alpha v(t) - \mu \cdot Sgn(v(t)) - y(t)).$$

Since there may exist locally two solutions, if both of them exist we have to chose that one wich better reflects the behaviour of the real-world system modeled by the given equation. We propose to use and compare experimentally several choice functions. In the following numerical experiments we have put $u(t) = \sin(t)$, $\alpha = 1$, $\mu = 0.4$, $\varepsilon = 0.0001$, $\delta = 0.6$. These choice functions are:

- First is choosing allways the "old" Hukuhara differentiable solution. Surely this is the most inconvenient choice, since uncertainty is allways increasing and in this case the uncertainty increases exponentially. This fact is illustrated in Fig. 1.

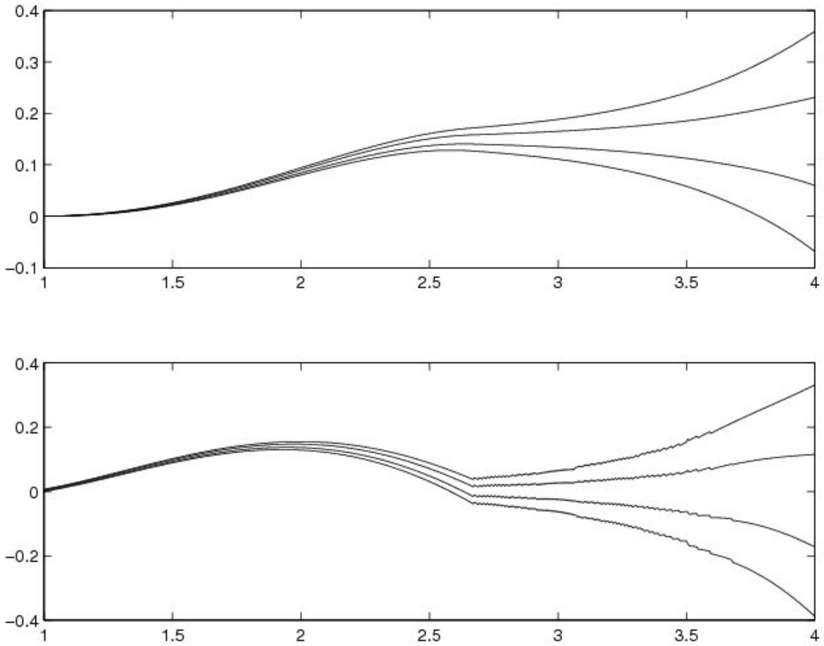

Figure 1

Hukuhara differentiable solution of the dry friction equation

- Second is choosing solutions with increasing support if the "core", i.e. midpoint of the 1-level set is increasing in absolute value (this choice is based on the hypothesis that the uncertainty increases together with the value). In our model this is not consistent with the usual real behavior of the velocity. That is the static friction appears at velocity 0 and in this case around zero the uncertainty should increase. The solution of the FDE modeling dry friction is in this case presented in Fig. 2.

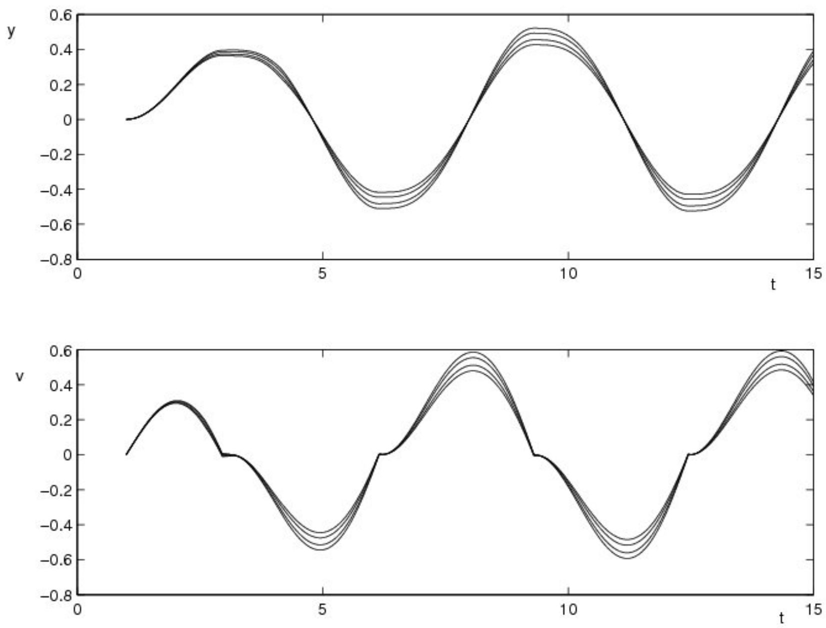

Figure 2

Solution under the second choice function

- Third possible choice is that we select the solution with decreasing uncertainty, whenever it exists. That is we are searching for solutions which are as certain as possible. In this case we have obtained almost certain solutions. In our opinion this is not realistic.
- Next is related to the previous possible choice function with the suplementary condition that we would like to maintain the uncertainty greater than a threshold value. The solution obtained this way is shown in Fig. 3.

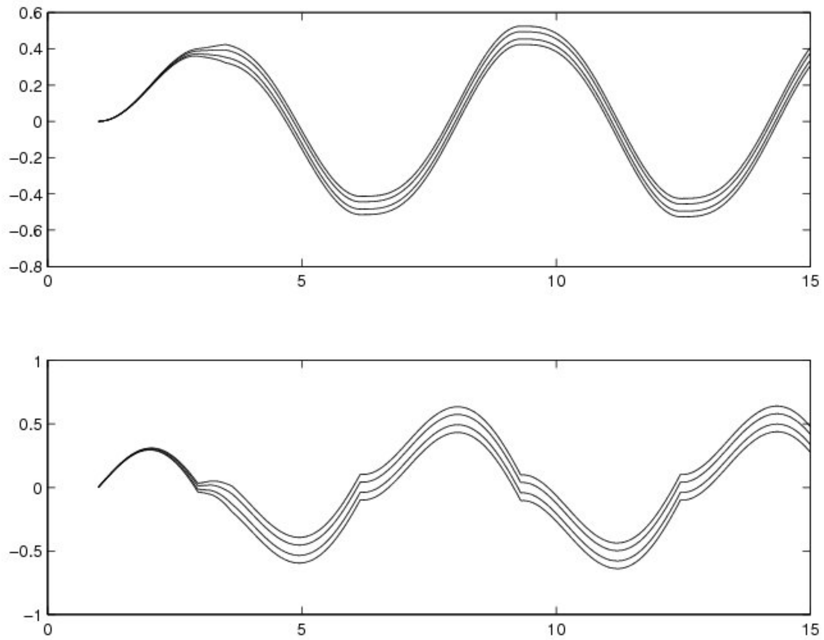

Figure 3
Solution for which uncertainty do not decrease below a threshold

- Again we choose if possible the solutions with decreasing uncertainty but we set a threshold value for the velocity, under which its uncertainty increases. Also, the uncertainty on the coordinate is not allowed to go below a threshold value. This choice is motivated by the physical properties of the system. The behavior of the solution is represented in Fig. 4.

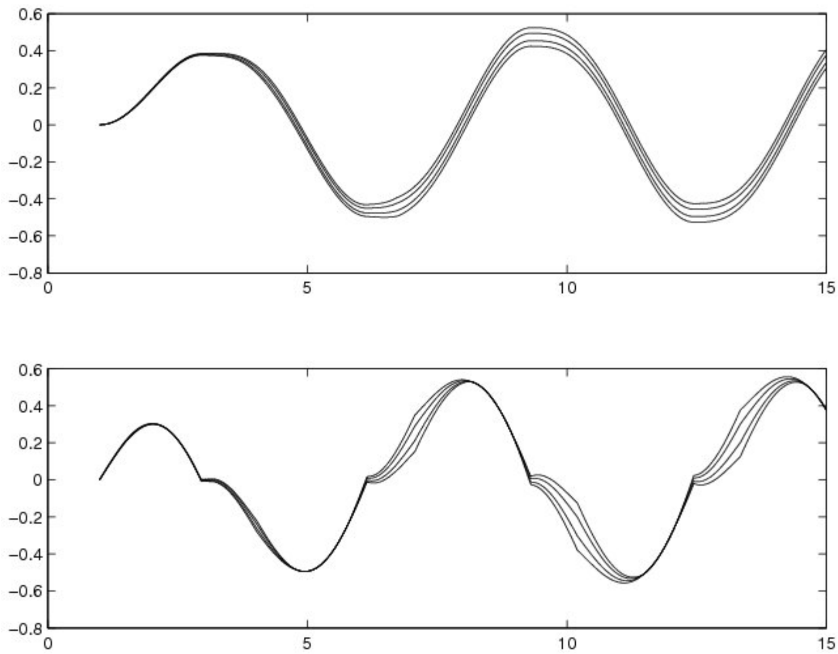

Figure 4

Solution with the assumption that small velocity implies increasing uncertainty

5 The Fuzzy Differential Equations of the Hydraulic Differential Servo Cylinder

For modeling the operation of the cylinder we use fuzzy differential equations since the friction term as discussed in a previous section can be regarded as a fuzzy number. If y denotes the linear position of the piston then its acceleration is determined by the fuzzy differential equation

$$y'' = \frac{1}{m}\left[\left(p_A - \frac{1}{\varphi} p_B\right) A_A - F_f(v) - F_d\right], \tag{3}$$

where p_A and p_B denote the pressures in chamber A and B of the piston, φ is the ratio of the active surfaces of the appropriate sides of the piston A_A and A_B,

m is the mass of the piston, F_f denotes the force of the internal friction between the piston and the cylinder, F_d denotes the external disturbance forces.

The fuzzy friction term F_f is given by

$$F_f = \mu v + Sgn(v)\left(F_c + F_s e^{-\frac{|v|}{\alpha}}\right),$$

where $Sgn(v) = Sgn_{\varepsilon,\delta}(v)$ is as in (1).

The oil-pressures in the chambers also depends on the piston's position and velocity, so these are modeled again by fuzzy differential equations

$$p_A' = \frac{E_o}{V_A(y)}\left(-A_A \cdot v + B_v K_v \cdot a(p_A, U) \cdot U\right)$$
$$p_B' = \frac{E_o}{V_B(y)}\left(-\frac{A_A}{\varphi} \cdot v + B_v K_v \cdot a(p_B, -U) \cdot U\right),$$

where E_{oil} describes the elasticity of the working fluid establishing relationship between the relative volumetric compression and the increase in the fluid pressure, B_v denotes the flow resistance, K_v is the valve amplification, U is the normalized valve voltage. The oil volumes in the pipes and the chambers are expressed as

$$V_A(y) = V_{pipe} + A_A \cdot y$$
$$V_B(y) = V_{pipe} + A_A \cdot (H - y).$$

The functions a contains again the signum function in its expression

$$a(p,U) = R\left(\frac{p_0 + p_t - 2p}{2} Sgn(U) + \frac{p_0 - p_t}{2}\right),$$

where $R(x) = sign(x)\sqrt{x}$, here $sign$ denotes the usual crisp signum function and $Sgn(U)$ is a fuzzy term given as in (1).

Conclusions and Further Research

We have proposed a fuzzy model for dry friction and we have performed numerical experiments on it. These experiments show that the fuzzy model is more realistic that the crisp one. Indeed, the experiments show that if we set the parameter $\varepsilon = 0$ (term which determines the fuzzy signum function), that is if we consider the friction a crisp phenomenon then we have obtained solutions for which uncertainty is increasing exponentially. Also, if we set $\delta = 1$, that is we have an interval valued friction as proposed in [5], then in our experiments we did not obtained a stable solution. Since the experience shows that however the solution of the system is subject of uncertainty the velocity cannot be ∞, we may suppose that our model is more realistic than crisp models.

The behavior of the fuzzy model motivated us to propose in this paper a fuzzy model of a hydraulic differential cylinder. The fuzzy differential equations which are building our model can be solved in a similar way as the dry friction equations in the previous sections. The experimental study of the presented model as well as its experimental validation is subject of future research.

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