Optimal Control of the Three Tank System in H_2/H_∞ Space

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Abstract: The three tank system is a widely used laboratory system in control theory. In this case study an optimal control method is presented over it. Building the linearized mathematical model, the H_2/H_∞ minimax method is applied, known also as minimax, extended or disurbance rejection LQ control, and the obtained results are compared with the classical LQ control, proving the superiority of the minimax LQ mehtod. The simulations were carried out in Matlab-Simulink.

Keywords: three tank system, disturbance rejection, LQ control, minimax control

1 Introduction

Liquid level control has a very large application domain in industry. Its most representative didactical equipments are the tank systems, i.e. one, three [1] or four tank systems [2]. Moreover, the three tank system (3TS) is one of the most widely used laboratory system in control theory.

This case-study deals with the control of the 3TS using modern control methods. More precisely, the extended LQ control, known as disturbance rejection LQ control is applied [3], and the obtained results are compared with those obtained by the classical LQ one. The simulations were carried out in Matlab-Simulink.

2 Mathematical Modeling of the 3TS

The basic equation of the mathematical model is based on Bernoulli's law for liquids [4], [5], [6]. The system is a MIMO system with two inputs and two outputs presented in Figure 1.

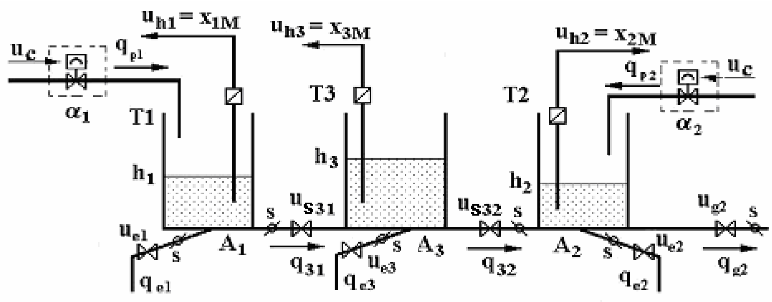

Figure 1
General model of the 3TS

The input voltage signals of the pumps, u_{c1} and u_{c2} were considered the inputs of the system, while the liquid levels h_1 and h_2 the outputs. The disturbances of the system were considered the positions of the valves (0 = closed, 1 = open), namely the valves between the tanks, u_{s13}, u_{s32}, the nominal outflow valve u_{g2} and the outflow valve of each tank, u_{e1}, u_{e2}, u_{e3}. The linearized mathematical model given by equations (1)-(5) was obtained from the non-linear model, [4], [5], using linearization over trajectories in the vicinity of stationary variables marked with indices '$_0$'.

One can observe the disadvantage of the obtained mathematical model, namely that the equations are modelling correctly the 3TS only when the liquid levels are not the equal. Therefore, the paper discusses only the situation when the liquid levels are different.

$$\begin{aligned}\frac{d\Delta h_1}{dt} = &-\frac{\sqrt{2g}}{A}.sgn(h_{10}-h_{30})\sqrt{|h_{10}-h_{30}|}.\Delta u_{s1} - \\ &-\frac{\sqrt{2g}}{A}u_{s10}.sgn(h_{10}-h_{30})\frac{1}{2\sqrt{|h_{10}-h_{30}|}}.\Delta h_1 + \\ &+\frac{\sqrt{2g}}{A}u_{s10}\frac{1}{2\sqrt{|h_{10}-h_{30}|}}sgn(h_{10}-h_{30}).\Delta h_3 - \\ &-\frac{\sqrt{2g}}{A}\sqrt{h_{10}}.\Delta u_{e1} - \frac{\sqrt{2g}}{A}u_{e10}\frac{1}{2\sqrt{h_{10}}}.\Delta h_1 + \frac{c_1}{A}.\Delta u_{c1}\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{d\Delta h_2}{dt} = & -\frac{\sqrt{2g}}{A}\sqrt{h_{20}}.\Delta u_{e2} - \frac{\sqrt{2g}}{A}u_{e20}\frac{1}{2\sqrt{h_{20}}}.\Delta h_2 + \\ & +\frac{\sqrt{2g}}{A}sgn(h_{30}-h_{20}).\sqrt{|h_{30}-h_{20}|}.\Delta u_{s2} - \\ & -\frac{\sqrt{2g}}{A}u_{s20}.sgn(h_{30}-h_{20}).\frac{1}{2\sqrt{|h_{30}-h_{20}|}}.\Delta h_2 - \\ & -\frac{\sqrt{2g}}{A}\sqrt{h_{20}}.\Delta u_{g2} - \frac{\sqrt{2g}}{A}u_{g20}\frac{1}{2\sqrt{h_{20}}}.\Delta h_2 + \\ & +\frac{\sqrt{2g}}{A}u_{s20}.sgn(h_{30}-h_{20}).\frac{1}{2\sqrt{|h_{30}-h_{20}|}}.\Delta h_3 + \frac{c_2}{A}.\Delta u_{c2}\end{aligned} \quad (2)$$

$$\begin{aligned}\frac{d\Delta h_3}{dt} = & \frac{\sqrt{2g}}{A}.sgn(h_{10}-h_{30})\sqrt{|h_{10}-h_{30}|}.\Delta u_{s1} + \\ & +\frac{\sqrt{2g}}{A}u_{s10}.sgn(h_{10}-h_{30}).\frac{1}{2\sqrt{|h_{10}-h_{30}|}}.\Delta h_1 - \\ & -\frac{\sqrt{2g}}{A}u_{s10}.sgn(h_{10}-h_{30}).\frac{1}{2\sqrt{|h_{10}-h_{30}|}}.\Delta h_3 - \\ & -\frac{\sqrt{2g}}{A}sgn(h_{30}-h_{20}).\sqrt{|h_{30}-h_{20}|}.\Delta u_{s2} + \\ & +\frac{\sqrt{2g}}{A}u_{s20}.sgn(h_{30}-h_{20}).\frac{1}{2\sqrt{|h_{30}-h_{20}|}}.\Delta h_2 - \\ & -\frac{\sqrt{2g}}{A}u_{s20}.sgn(h_{30}-h_{20}).\frac{1}{2\sqrt{|h_{30}-h_{20}|}}.\Delta h_3 - \\ & -\frac{\sqrt{2g}}{A}.\sqrt{h_{30}}.\Delta u_{e3} - \frac{\sqrt{2g}}{A}u_{e30}.\frac{1}{2\sqrt{h_{30}}}.\Delta h_3\end{aligned} \quad (3)$$

$$\Delta y1 = \Delta h1 \quad (4)$$

$$\Delta y2 = \Delta h2 \quad (5)$$

In the case when the liquid levels are equal, the 3TS becomes a collection of three independent one tank system, due to the fact that they are in equilibrium.

Comparing the non-linear and linearized system (the steady-state points were selected h_{10} = 50 *cm*, h_{20} = 10 *cm*, and h_{30} = 30 *cm*), the linearized system proved to approximate correctly the non-linear system, [5].

3 Minimax Control of the 3TS

In this situation the case study was considered with every valve in open position:

$$u_{s13} = u_{s32} = u_{g2} = u_{e1} = u_{e2} = u_{e3} = 1 \tag{6}$$

The initial liquid levels were the same as before: $h_{10} = 50\ cm$, $h_{20} = 10\ cm$, and $h_{30} = 30\ cm$.

The disturbance rejection problem is based on a minimax criteria [1], where the system dynamics is generally described by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ld(t) \\ y(t) &= Cx(t) \end{aligned} \tag{7}$$

Here one can observe that the desired input (control signal, $u(t)$) is treated separately from the disturbance input $d(t)$. As a result the general B matrix of an LTI system is separated in two: in a new B matrix (for the control signal) and an L matrix (for the disturbance signal).

The quadratic cost functional is extended from the classical LQ problem to include the effect of disturbance $d(t)$ explicitly:

$$J(u,d) = \frac{1}{2}\int_0^\infty [y^T(t)y(t) + u^T(t)u(t) - \gamma^2 d^T(t)d(t)]\,dt \tag{8}$$

It can be seen that the R and Q matrix from the classical LQ control description are not appearing now and the weighting is made here by one parameter, γ.

(8) reflects that the disturbance attempts to maximize the cost while we want to find a control $u(t)$ that minimizes the maximum cost achivable by the disturbance (by the worst case disturbance).

According to [1], the optimal control and the worst case disturbance is given by:

$$u^*(t) = -B^T P x^*(t) \tag{9}$$

$$d^*(t) = \frac{1}{\gamma^2} L^T P x^*(t) \tag{10}$$

where P is a positive definite symmetric solution to the modified control algebraic Riccati equation (MCARE):

$$PA + A^T P + C^T C - P(BB^T - \frac{1}{\gamma^2} LL^T)P = 0 \tag{11}$$

To solve the above formulated problem for the 3TS firstly we had to determine with Matlab the optimal γ ($\gamma \in [0,1]$). Using γ-iteration it has resulted for the optimal value: γ_{min}=0.0115, [6].

To check the optimality of the obtained result, the disturbance rejection was tested in case of the 3TS for three values of γ_{min}, namely: $\gamma = \gamma_{min}$, $\gamma = 10\ \gamma_{min}$, $\gamma = 100\ \gamma_{min}$.

The resulted solutions of the MCARE are given by Table 1.

Table 1
The solutions of MCARE for different γ

γ	**P**		
γ_{min}	278.2956	-7.6472	13.6618
	-7.6472	13.4040	-0.2914
	13.6618	-0.2914	1.0330
10 γ_{min}	9.9723	-.00019	0.1562
	-0.0019	9.2368	0.1387
	0.1562	0.1387	0.0562
100 γ_{min}	9.5114	0.0018	0.1435
	0.0018	9.0144	0.1332
	0.1435	0.1332	0.0529

In the same time a classical LQ design was made too (without γ), [6]. The obtained solution was:

$$P = \begin{pmatrix} 9.4641 & 0.0021 & 0.1422 \\ 0.0021 & 8.9907 & 0.1327 \\ 0.1422 & 0.1327 & 0.0525 \end{pmatrix} \tag{12}$$

One can see that this is very similar to the case of disturbance rejection LQ for 100 γ_{min}. The so obtained result demonstrates once again the theory, that for big values of γ, the disturbance rejection LQ method becomes again a classical LQ problem.

The illustration of this affirmation is presented applying step disturbances on the 3TS. The change of the liquid level (in meters) was simulated in time (seconds).

In Figure 2 the biggest possible change of h_2 is presented for each γ-case considered, over each disturbance effect (u_{s13}, u_{s32}, u_{g2}, u_{e1}, u_{e2}, u_{e3}).

It can be seen that for $\gamma_{min} = G = 1$ the biggest disturbance can be applied for u_{s13}, u_{e1}, u_{e3} (for Tank 1 and Tank 3). However for u_{s32}, u_{g2}, u_{e2} the effect of the disturbances were minimized on h_2, as these values are the characteristic disturbances for Tank 2 (and now the disturbance was considered over h_2). More details can be obtained from [6].

From the first output (h_1) point of view, similar results were obtained, but now the disturbances are maximized for u_{s32}, u_{g2}, u_{e2}, u_{e3} and are minimized for u_{s13}, u_{e1} (Figure 4).

Figure 3 presents the results of simulating the effect of step disturbance over a classical LQ problem, [6]. It can be seen that the results in Figure 2 (for 100 γ_{min} (*G*=100)) and Figure 3 are the same.

The same conclusion can be seen comparing the results of Figure 4 with Figure 5.

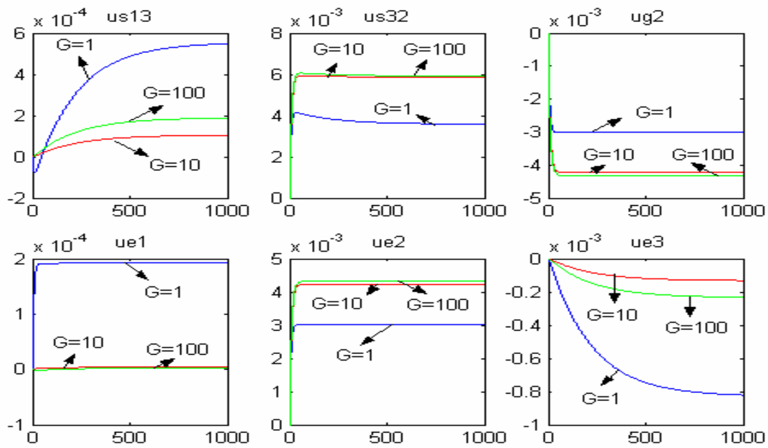

Figure 2
Disturbance rejection over h_2 for different γ values in time (G-multiple of γ_{min})

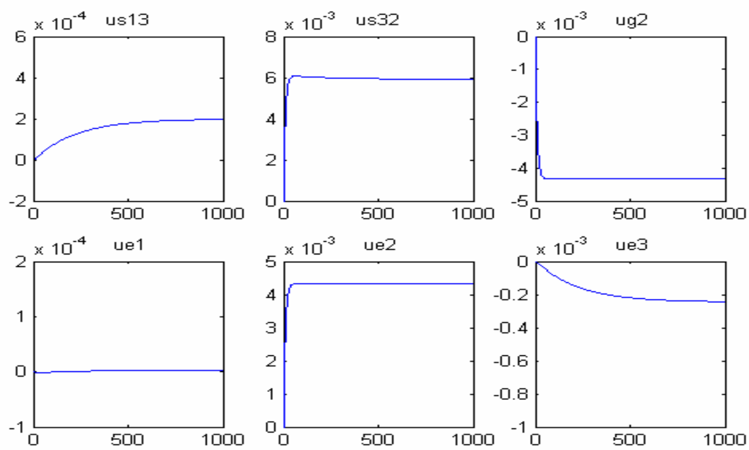

Figure 3
Disturbance rejection over h_2 in time for classical LQ

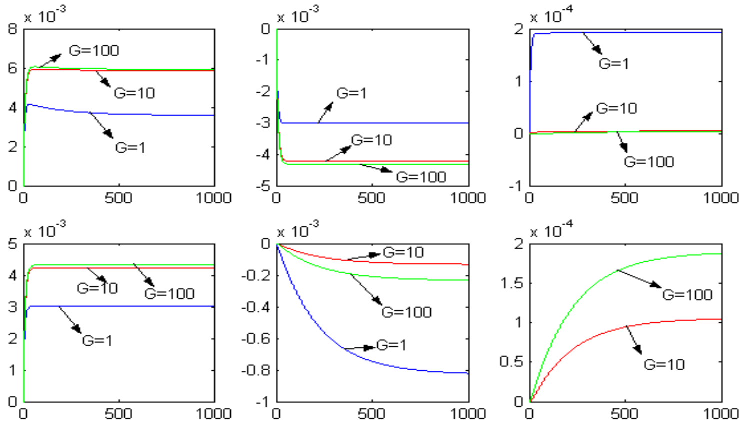

Figure 4

Disturbance rejection over h_1 for different γ values in time (G-multiple of γ_{min})

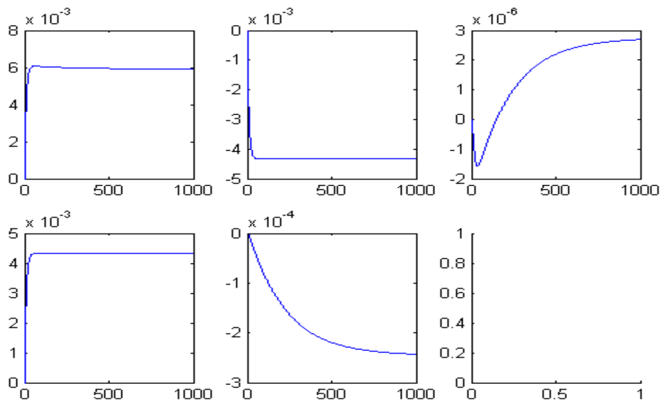

Figure 5

Disturbance rejection over h_1 in time for classical LQ

Conclusions

In this paper the disturbance rejection LQ control was applied on the 3TS and the obtained results were compared with those obtained by the classical LQ method. The superiority of the minimax control was demonstrated, namely that for big weighting values obtained for the worst case disturbance, the disturbance rejection LQ method becomes a classical LQ problem.

Acknowledgement

This research has been supported by Hungarian National Scientific Research Foundation, Grant No. OTKA T042990 and by Hungarian National Office for Research and Technology (RET-04/2004).

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