Pseudo-Fourier Transform

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Abstract: In signal processing applications, a key role is played by the Fourier transforms. The mathematical structure underlying Fourier analysis is the linear structure. In the present paper we propose and study a tool designed for nonlinear signal and image processing, that is novel Fourier-like transforms, based on a pair of pseudo-operations consisting of a uninorm and an absorbing norm. We analyse in the present paper theoretical properties of the proposed pseudo-Fourier transform.

Keywords: pseudo-operations, Fourier transform

1 Introduction

In signal processing, if we regard any of the classical results, it can be observed that the underlying algebraic structure is the linear space structure. Naturally, the following question raises: is the linear space structure the only one which can be used for signal processing purposes? Moreover, are the addition and multiplication of the reals the only operations that can be used for defining signal transformation techniques? Do all operators used in the signal and image processing need to be linear?

In several research fields, idempotent analysis and pseudo-linear structure have already shown their power in dealing with problems where linear structure is not helpful ([6, 8, 11, 12, 13, 14, 16]). This pseudo-linear structure is based on a so called pseudo-addition and a pseudo-multiplication. The mathematical apparatuses given on this structure are known as idempotent analysis ([6, 8]) and pseudo-analysis (see [10, 11, 12, 13, 14, 15]).

The same question as above was proposed in Approximation Theory, i.e., is the linear structure the only one that can be used in the classical Approximation Theory? The answer was also negative and, in this sense, maxitive Shepard-like approximation operators were proposed in [3]. Also, a discrete cosine transform method based on a pair consisting of a generated uninorm and an absorbing norm was proposed (see [2]) and studied mainly from the image processing point of view.

It is worth of noticing that the used operations do not need to form only the pseudo-linear structures. As shown in [17], various algebraical structures are at disposal. There the introduced approximation operators over a BL-algebra proved to be an effective tool for the data compression [18] as well as for the image coding and decoding problem [19]. Moreover, this investigation is closely related to the theory of fuzzy sets and systems, hence, it widen out the area of possible applications.

In the present paper we continue this line of research and we propose a pseudo-linear analogous of the classical Fourier Transform.

The proposed pseudo-Fourier transforms are defined based on a pair consisting of a generated uninorm and absorbing norm. Uninorms were introduced by Yager and Rybalov [20] as a generalization of t-norms and t-conorms ([5]). For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval. Absorbing norms are a generalization of the well-known median. For the absorbing norm (see [1]), a given element has absorbing property, i.e., its composition with any other element gives the absorbing element itself. We will use in the present paper a uninorm based on an additive generator and an absorbing norm based on the same (but multiplicative) generator. For such a pair of operations the distributivity property holds.

After the preliminary section, i.e., in Section 3, we define and study the pseudo-Fourier transform. In Section 4 we present the Inverse pseudo-Fourier transform. At the end of the paper some conclusions and topics for further research are given.

2 Preliminary notions

As already mentioned, construction presented in this paper has been done in the pseudo-analysis' framework. It is based on a special type of generated semiring, i.e. on a special type of semiring of the second class.

Let $[a, b]$ be closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be total order on $[a, b]$. Structure $([a, b], \oplus, \odot)$ is a *semiring* if the following hold:

- $\oplus$ is *pseudo-addition*, i.e., a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by $\mathbf{0}$;

- $\odot$ is *pseudo-multiplication*, i.e., a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively non-decreasing, associative and with a unit element denoted by $\mathbf{1}$;
- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Semirings with continuous (up to some points) pseudo-operations are divided into three classes. The first class contains semirings with idempotent pseudo-addition and non idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by monotone and continuous generator function form the second class, and semirings with both idempotent operations belong to the third class. More on this structure as well as on measures and integrals constructed on it can be found in [6, 7, 8, 9, 10, 11, 12].

For the purpose of this construction we shall consider a semiring of the second class on the unite interval, i.e. $([0,1], \oplus, \odot)$. As a pseudo-addition $\oplus : [0,1]^2 \to [0,1]$, the representable uninorm with neutral element $\mathbf{0} = e \in (0,1)$ will be used. In this case, for given $e \in (0,1)$ and a strictly increasing continuous function $g : [0,1] \to \overline{\mathbb{R}}$ such that $g(0) = -\infty$, $g(e) = 0$ and $g(1) = +\infty$, operation $\oplus$ is

$$x \oplus y = g^{-1}(g(x) + g(y)), \tag{1}$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$. If $(x,y) \in \{(0,1),(1,0)\}$, one of the following conventions will be accepted: either $0 \oplus 1 = 1 \oplus 0 = 0$ or $0 \oplus 1 = 1 \oplus 0 = 1$.

Remark 1 *A uninorm $U : [0,1]^2 \to [0,1]$ is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$ (see [20]). Specially, for $e = 1$ this type of a operator is a triangular norm, and for $e = 0$ a triangular conorm (see [5]). The class of representable uninorms with a neutral element $e \in (0,1)$, i.e., the class used in this paper, has been characterized in [4].*

Now, corresponding pseudo-multiplication $\odot$ is

$$x \odot y = g^{-1}(g(x)g(y)), \tag{2}$$

and, for previously described generating function g, it belongs to the class of so-called absorbing norms.

Remark 2 *An absorbing norm $\odot : [0,1]^2 \to [0,1]$ is a commutative, associative and increasing binary operator with an absorbing element $a \in [0,1]$, i.e. $(\forall x \in [0,1])(x \odot a) = a$). ([1])*

Some of the basic properties of operations (1) and (2) are (see [2]):

(i) $\odot$ is an absorbing norm with e as absorbing element;

(ii) $\mathbf{1} = g^{-1}(1)$;

(iii) for all $x \in (0,1)$ there exists $\ominus x \in (0,1)$ such that $x \oplus (\ominus x) = \mathbf{0}$.

Since $\ominus x = g^{-1}(-g(x))$ ([2]), the pseudo-subtraction for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ can be given in the following form:

$$x \ominus y = x \oplus (\ominus y) = g^{-1}\left(g(x) - g(y)\right). \tag{3}$$

3 Pseudo-Fourier transform

Let $([0,1], \oplus, \odot)$ be a previously described semiring, where $\oplus$ and $\odot$ are given by generating function $g : [0,1] \to \overline{\mathbb{R}}$. It can be easily shown that any function $f : \mathbb{R} \to [0,1]$ can be split into two parts with respect to $\oplus$, i.e.,

$$f(x) = E_p(x) \oplus O_p(x),$$

where

$$E_p(x) = g^{-1}\,(1/2) \odot (f(x) \oplus f(-x)) \quad \text{and} \quad O_p(x) = g^{-1}\,(1/2) \odot (f(x) \ominus f(-x)),$$

and $\ominus$ is operation given by (3). Of special interest for this paper are facts that E_p and $g \circ E_p$ are even functions and $g \circ O_p$ is an odd function.

Definition 3 The pseudo-Fourier cosine transform *based on the semiring* $([0,1], \oplus, \odot)$ *of a measurable function* $f : \mathbb{R} \to [0,1]$ *is*

$$\mathcal{F}_C^{\oplus}[f(x)](\omega) = g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right) \odot \int_{[-\infty,\infty]}^{\oplus} g^{-1}\left(\cos\left(\omega x\right)\right) \odot E_p(x)\, dx, \tag{4}$$

for every real number ω *(if the right side exists).*
The pseudo-Fourier sine transform *based on the semiring* $([0,1], \oplus, \odot)$ *of a measurable function* $f : \mathbb{R} \to [0,1]$ *is*

$$\mathcal{F}_S^{\oplus}[f(x)](\omega) = g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right) \odot \int_{[-\infty,\infty]}^{\oplus} g^{-1}\left(\sin\left(\omega x\right)\right) \odot O_p(x)\, dx, \tag{5}$$

for every real number ω *(if the right side exists).*

Remark 4 *Integrals on the right in (4) and (5) are pseudo-integrals based on the given semiring of the second class, i.e., g-integrals (see [7, 9, 11]). Definition of g-integral gives us following forms of transforms (4) and (5):*

$$\mathcal{F}_C^{\oplus}[f(x)](\omega) = g^{-1}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g\left(E_p(x)\right)\cos\left(\omega x\right)\, dx\right)$$

and

$$\mathcal{F}_S^{\oplus}[f(x)](\omega) = g^{-1}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g\left(O_p(x)\right)\sin\left(\omega x\right)\, dx\right).$$

As in the classical case, the pseudo-Fourier transform of some measurable function is expressed in terms of the pseudo-Fourier cosine transform and pseudo-Fourier sine transform.

Definition 5 The pseudo-Fourier transform *based on the semiring* $([0,1],\oplus,\odot)$ *of a measurable function* $f:\mathbb{R}\to[0,1]$, *for every real number* ω, *is*

$$\mathcal{F}^{\oplus}[f(x)](\omega) = \mathcal{F}_C^{\oplus}[f(x)](\omega) - i\mathcal{F}_S^{\oplus}[f(x)](\omega), \tag{6}$$

where $\mathcal{F}_C^{\oplus}$ *and* $\mathcal{F}_S^{\oplus}$ *are transforms given by (4) and (5).*

Some of the basic properties of this Fourier-type transform that can be easily shown are:

- pseudo-linearity, i.e., if $f,g:\mathbb{R}\to[0,1]$ are measurable functions and a and b are some real parameters, then

  $$\mathcal{F}^{\oplus}(a\odot f) = a\odot\mathcal{F}^{\oplus}(f)$$

  and

  $$\mathcal{F}^{\oplus}(f\oplus h) = \left(\mathcal{F}_C^{\oplus}(f)\oplus\mathcal{F}_C^{\oplus}(h)\right) - i\left(\mathcal{F}_S^{\oplus}(f)\oplus\mathcal{F}_S^{\oplus}(h)\right);$$

- pseudo-shift property, i.e.,

  $$\mathcal{F}_C^{\oplus}[f(x-a)](\omega) = g^{-1}\left(\cos a\omega\right)\odot\mathcal{F}_C^{\oplus}[f(x)](\omega)\ominus g^{-1}\left(\sin a\omega\right)\odot\mathcal{F}_S^{\oplus}[f(x)](\omega)$$

  and

  $$\mathcal{F}_S^{\oplus}[f(x-a)](\omega) = g^{-1}\left(\cos a\omega\right)\odot\mathcal{F}_S^{\oplus}[f(x)](\omega)\oplus g^{-1}\left(\sin a\omega\right)\odot\mathcal{F}_C^{\oplus}[f(x)](\omega)$$

  where $f:\mathbb{R}\to[0,1]$ is measurable function and a some real parameter;

- if $f:\mathbb{R}\to[0,1]$ is measurable and bounded (in sense that $\lim_{x\to\pm\infty} f(x) = \mathbf{0}$ with respect to some pseudo-metric based on generator g) function, then

  $$\mathcal{F}_C^{\oplus}\left[\frac{d^{\oplus}f(x)}{dx}\right](\omega) = g^{-1}(\omega)\odot\mathcal{F}_S^{\oplus}[f(x)](\omega)$$

  and

  $$\mathcal{F}_S^{\oplus}\left[\frac{d^{\oplus}f(x)}{dx}\right](\omega) = -g^{-1}(\omega)\odot\mathcal{F}_C^{\oplus}[f(x)](\omega),$$

  where $\dfrac{d^{\oplus}}{dx}$ is pseudo-derivative (see [10, 11]).

Also, following pseudo-convolution theorem can be easily proven.

Theorem 6 *If $\star$ is pseudo-convolution of the second type based on given semiring $([0,1], \oplus, \odot)$ and $f, g : \mathbb{R} \to [0,1]$ are measurable functions, then*

$$\mathcal{F}_C^{\oplus}(f \star h) = g^{-1}\left(\sqrt{2\pi}\right) \odot \left(\mathcal{F}_C^{\oplus}(f) \odot \mathcal{F}_C^{\oplus}(h) \ominus \mathcal{F}_S^{\oplus}(f) \odot \mathcal{F}_S^{\oplus}(h)\right)$$

and

$$\mathcal{F}_S^{\oplus}(f \star h) = g^{-1}\left(\sqrt{2\pi}\right) \odot \left(\mathcal{F}_C^{\oplus}(f) \odot \mathcal{F}_S^{\oplus}(h) \oplus \mathcal{F}_S^{\oplus}(f) \odot \mathcal{F}_C^{\oplus}(h)\right)$$

Proof is based on properties of classical Fourier transform and properties of pseudo-convolution of the second type given on g-semiring. More on pseudo-convolutions can be found in [15].

4 Inverse Pseudo-Fourier Transform

In the previous section, we have seen that some certain operation with functions corresponds to operations with their pseudo-Fourier images. The usefulness of this relationship becomes clear if we are able to convert from pseudo-Fourier images back to functions, i.e. if there exists the inverse transformation.

The simple calculation leads to

$$\begin{aligned} F(\omega) =& \mathcal{F}(g \circ f)(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} g \circ f(t) e^{-i\omega t}\, dt \\ =& \frac{1}{\sqrt{2\pi}} \left(\int\limits_{-\infty}^{\infty} g \circ E_p(t) \cos\, \omega t\, dt - i \int\limits_{-\infty}^{\infty} g \circ O_p(t) \sin\, \omega t\, dt \right) \\ =& g \circ \mathcal{F}_C^{\oplus}(f)(\omega) - i\, g \circ \mathcal{F}_S^{\oplus}(f)(\omega), \end{aligned}$$

where $\mathcal{F}$ is the classical Fourier transform. Observe that both $\mathcal{F}_C^{\oplus}(f)(\omega)$ and $g \circ \mathcal{F}_C^{\oplus}(f)(\omega)$ remain even and $g \circ \mathcal{F}_S^{\oplus}(f)(\omega)$ is odd.

For simplicity, let us use the following notation:

$$\int_{\mathbb{R}}^{\oplus} p(x) \odot dm_q(x) = g^{-1}\left(\frac{1}{\sqrt{2\pi}}\right) \odot \int_{[-\infty,\infty]}^{\oplus} p(x) \odot q(x)\, dx$$

and

$$\int_a^b p(x)\, d(x) = \frac{1}{\sqrt{2\pi}} \int_a^b p(x)\, dx.$$

Now, the inverse transform, obtained in analogy with the classical approach, is of the following form:

$$\begin{aligned}\mathcal{F}_{\oplus}^{-1}\{\mathcal{F}^{\oplus}(f)\}(x) = &\int_{[-\infty,\infty]}^{\oplus} g^{-1}(\cos\, tx) \odot dm_{\mathcal{F}_C^{\oplus}(f)}(t) \\ &\oplus \int_{[-\infty,\infty]}^{\oplus} g^{-1}(\sin\, tx) \odot dm_{\mathcal{F}_S^{\oplus}(f)}(t) \\ &= \mathcal{F}_{\oplus,C}^{-1}\{\mathcal{F}_C^{\oplus}(f)\}(x) \oplus \mathcal{F}_{\oplus,S}^{-1}\{\mathcal{F}_S^{\oplus}(f)\}(x) \quad (7)\end{aligned}$$

Further, in order to reconstruct the original function from its even and odd parts, we can ask whether $\mathcal{F}_{\oplus,C}^{-1}\{\mathcal{F}_C^{\oplus}(f)\} = E_p$ and $\mathcal{F}_{\oplus,S}^{-1}\{\mathcal{F}_S^{\oplus}(f)\} = O_p$. The following Theorem will give us an answer. We will start with the preliminary Lemmas.

Lemma 7 *If $g \circ f \in L^1$ then $\mathcal{F}^{\oplus}(f)$ is continuous.*

Lemma 8 *Let*

$$h_\lambda(x) = \int_0^\lambda \cos(tx)\, d(t) \quad \text{and} \quad (f * h)(x) = g^{-1} \int_{-\infty}^{\infty} g \circ f(x-y) h(y)\, d(y).$$

If $g \circ f \in L^1$ then

$$(f * h_\lambda)(x) = \int_{[0,\lambda]}^{\oplus} g^{-1}(\cos\, tx) \odot dm_{\mathcal{F}_C^{\oplus}(f)}(t) \oplus \int_{[0,\lambda]}^{\oplus} g^{-1}(\sin\, tx) \odot dm_{\mathcal{F}_S^{\oplus}(f)}(t). \quad (8)$$

Theorem 9 *If $g \circ f \in L^1$ and $g \circ \mathcal{F}_S^{\oplus}(f), g \circ \mathcal{F}_C^{\oplus}(f) \in L^1$ then*

$$\lim_{\lambda\to\infty} (f * h_\lambda)(x) = \mathcal{F}_{\oplus}^{-1}\{\mathcal{F}^{\oplus}(f)\}(x) = f(x), \quad (9)$$

nearly everywhere and $\mathcal{F}_{\oplus}^{-1}\{\mathcal{F}^{\oplus}(f)\}(x)$ is continuous.

The proof of (9) is based on the limit translation of (8) for $\lambda \to \infty$ and the continuity of $\mathcal{F}_{\oplus}^{-1}\{\mathcal{F}^{\oplus}(f)\}(x)$ follows from Lemma 7.

5 Conclusion

The main aim of this paper has been to present one possible generalization, based on the pseudo-analysis' apparatus, of well known classical Fourier transform. Some further research of this problem should concern construction and investigation of discrete pseudo-Fourier transforms based on presented background, as well as possible applications of obtained transforms (both continuous and discrete), specially in the area of signal and image processing, but also in the study of nonlinear partial differential equations.

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