Application of Aggregation Operators in Solution of Nonlinear Equations

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Abstract: In this paper a modifications of Newton's iterative method for solving nonlinear equations based on the Aggregation operators: arithmetic, harmonic, geometric and finally the root-power mean has been considered. The convergence properties of considered methods have been analyzed. The convergence order for simple zeros is three and linear for multiple zeros. Required computational evaluations per iteration is three. By utilizing some properties of the root-power mean we suggest a most efficient method for finding the multiple zeros. Illustrative examples are given.

Key words and phrases: Newton's method, Aggregation operators, Iterativ methods, Order of convergence, Function evaluations.

1 Introduction

We consider a nonlinear equation

$$f(x) = 0, \quad f : D \subseteq R \to R \tag{1}$$

with real zero(root) α. The best know numerical method for solving equation (1) is the calssical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \tag{2}$$

where x_0 is an initial approximation sufficiently close to α.

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The iterative method for solving equation (1) produces the iterative sequence $\{x_0, x_1, \ldots\}$, such that $\lim_{n\to\infty} x_n = \alpha$.

The zero α is said to be a simple if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. If $f(\alpha) = f'(\alpha) = ... = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$ for $m > 1$ then the zero α is of multiplicity m.

The best know numerical method for solving equation (1) is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \tag{3}$$

where x_0 is an initial approximation sufficiently close to α.

Definition 1 *If the sequence $\{x_n\}$ tends to a limit α such that*

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^{\kappa}} = C$$

for some $C \neq 0$ and $\kappa \geq 1$, then the order of convergence of the sequence is κ, and C is know as the asymptotic error constant (AEC).

If $\kappa = 1$, $\kappa = 2$ or $\kappa = 3$, the convergence is said to be *linearly*, *quadratically* or *cubic*, respectively.

Let $e_n = x_n - \alpha$ be the error in the nth iterate of the method which produces the sequence $\{x_n\}$. Then, the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}) = O(e_n^p)$$

is called the *error equation*. The value of p is called the order of convergence of this method.

The convergence order of the classical Newton's method is quadratically for simple zeros and linearly for multiple zeros. In recent years in literature [2, 14, 8] are given some variant of Newton's methods, based on arithmetic, harmonic and geometric mean. In [9] we suggest the method based on root-power mean. Accordingly, all this methods is based on aggregation operators.

Aggregation operators like: triangular norms and conorms, uninorms, copules, weighted arithmetic means, ordered weighted arithmetic means and compensated operators, make a special class of aggregation operators. All these operators are detailed considered in different scientific papers and monographs, for example monograph of E. P. Klement, R. Mesiar and E. Pap is dedicated to triangular norms [7], the ordered weighted averaging operators are considered in edition of R. R. Yager i J. Kacprzyk [11], while copules are presented in monograph of R. B. Nelsen [12]. Many results connected with aggregation operations can be found in edition of T. Calva, G. Mayor and R. Mesiar [1].

We going to represent a definition, some examples and properties of aggregation operators.

Definition 2 *The* **aggregation operator** *is a function* $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *such that*

i) $A(x_1, ..., x_n) \le A(y_1, ..., y_n)$*, whenever* $x_i \le y_i$ *for all* $i \in \{1, ..., n\}$.

ii) $A(x) = x$ *for all* $x \in [0,1]$.

iii) $A(0, ..., 0) = 0$ *and* $A(1, ..., 1) = 1$.

Operators Π, as the operator of product, arithmetic mean M, Min, Max and operator A_c are all aggregation operators.

$$\Pi(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i \,, \qquad M(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i \,,$$

$$Min(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)\,, \qquad Max(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)\,,$$

$$A_c(x_1, \ldots, x_n) = \max(\,0,\, \min(\,1,\, c + \sum_{i=1}^{n}(x_i - c))),$$

where the operator $A_c : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is defined for all $c \in (0,1)$.

The weakest aggregation operator, in designation A_w, and the strongest aggregation operator A_s, are given by following:

$$\begin{array}{lll} (\forall\ n \ge 2) & (x_1, \ldots, x_n) \ne (1, \ldots, 1) \ : & A_w(x_1, \ldots, x_n) = 0 \\ (\forall\ n \ge 2) & (x_1, \ldots, x_n) \ne (0, \ldots, 0) \ : & A_s(x_1, \ldots, x_n) = 1. \end{array}$$

Aggregation operators between each other can be compared like functions with n-variables. For any aggregation operator A is satisfied:

$$A_w \le A \le A_s.$$

Also, the following is satisfied:

$$A_w \le \Pi \le Min \le M \le Max \le A_s.$$

Example 1 *Aggregation operator:* $W_\triangle : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *defined by*

$$W_\triangle(\, x_1\,,\, x_2,\, \ldots,\, x_n) = \sum_{i=1}^{n} w_{in} x_i\,,$$

is the so called weighted arithmetic operator associated with weighted triangle $\triangle$.

Weighted arithmetic means are continuous, idempotent, linear, additive and self-dual aggregation operators.

Example 2 *Let $f:[0,1]\to[-\infty,+\infty]$ continuous and stricly monoton function. The aggregation operator $M_f:\bigcup\limits_{n\in\mathbb{N}}[0,1]^n\to[0,1]$, which is given by*

$$M_f(x_1,x_2,\ldots,x_n)=f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n}f(x_i)\Big),$$

is called **quasi-arithmetic mean**.

The class of quasi-arithmetic means, root-power operators $M_p:\bigcup\limits_{n\in\mathbb{N}}[0,1]^n\to[0,1]$, $p\in(-\infty,0)\cup(0,+\infty)$ is obtained by applying the function $f_p:[0,1]\to[-\infty,+\infty]$, $f_p(x)=x^p$ such as:

$$M_p(x_1,x_2,\ldots,x_n)=\Big(\frac{1}{n}\sum_{i=1}^{n}x_i^p\Big)^{\frac{1}{p}}.$$

Marginal members of these classes are $M_0=G=M_{\log x}$, which is the geometric mean, while $M_\infty=Max$ and $M_{-\infty}=Min$ which are not in class of quasi-arithmetic means.

2 Definition of the methods

In following, we give the short description of methods based on aggregation operators.

At first we consider the method proposed by Traub(1964) [10] and rediscovered by Weerakoon et al.(2000) in [14]:

$$\begin{aligned}&x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'(v_{n+1})},\\&\text{where } v_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\ n=0,1,\ldots,\end{aligned}\tag{4}$$

which is, in contrast to Newton's method (3), instead of $f'(x_n)$ using arithmetic mean of $f'(x_n)$ and $f'(v_{n+1})$. Therefore, it call *arithmetic mean Newton's method* (AM).

If we use harmonic mean instead of the arithmetic mean in (4), we obtain the *harmonic mean Newton's method* (HM):

$$\begin{aligned}&x_{n+1}=x_n-\frac{f(x_n)(f'(x_n)+f'(v_{n+1}))}{2f'(x_n)f'(v_{n+1})},\\&\text{where } v_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\quad n=0,1,\ldots\end{aligned}\tag{5}$$

proposed by Traub(1964) and rediscovered by Özban(2004) in [2].

Leading by such admission Lukić et al.(2005) in [8] proposed the following scheme

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{sign(f'(x_0))\sqrt{f'(x_n)f'(v_{n+1})}}, \\ \text{where } v_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots \end{aligned} \tag{6}$$

which we call *geometric mean Newton's method* (GM).

The root-power mean M_p for values a and b is defined by

$$M_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}},$$

where p is a real number, $p \neq 0$. The operator M_p add up to the arithmetic and harmonic mean when $p = 1$ and $p = -1$ respectively. Similarly, M_p add up to the geometric mean when $p \to 0$, by equation $\lim\limits_{p\to 0} \sqrt[p]{\frac{a^p+b^p}{2}} = \sqrt{ab}$.

If we use the root-power mean instead of arithmetic mean in (4), we get the new scheme

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{sign(f'(x_0)) \cdot M_p(|f'(x_n)|, |f'(v_{n+1})|)}, \\ \text{where } v_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \ \ p \neq 0, \ \ n = 0, 1, \ldots \end{aligned} \tag{7}$$

which we call *root-power mean Newton's method* (RPM), proposed by Ralević et. al. in [9]. It is easy to see that the above mentioned methods can be obtained as a special cases of the root-power mean method, especially for $p = 1$, $p = -1$ and $p \to 0$ we obtain the AM, HM and GM methods, respectively. Hence, it is clear that it is sufficient to consider the convergence behavior of the root-power mean method.

3 Analysis of convergence

The next theorem ensure the third order convergence of the RPM method for all $p \neq 0$ in case when we approximate a simple zero.

Theorem 1 *[9] Let $f : D \subseteq R \to R$ for an open interval D. Assume that f has first, second and third derivatives on the interval D and f has a simple root in $\alpha \in D$. If x_0 is sufficiently close to α, then the RPM method, defined by (7), converges cubically and satisfies the following error equation:*

$$e_{n+1} = \frac{1}{2}\left((p+1)c_2^2 + c_3\right) e_n^3 + O(e_n^4), \tag{8}$$

where $e_n = x_n - \alpha$ and constants $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$ for $j = 1, 2, 3, \ldots$.

It is worth noting that the rate of convergence can be improve by an appropriate choice of parameter p, $p_{opt} = -\left(\frac{2f'''(\alpha)f'(\alpha)}{3(f''(\alpha))^2} + 1\right)$. For $p = p_{opt}$ the convergence order is at least four. This is only (for now) theoretical possibility, because p_{opt} depends on the zero α.

In the next theorem we give the error equation of RPM method for multiple zeros.

Theorem 2 *Let $f : D \subseteq R \to R$, where D is an open interval. Assume that f is sufficiently many times differentiable on the interval D and f has a multiple zero of multiplicity m $(m > 1)$ in $\alpha \in D$. If x_0 is sufficiently close to α, then the root-power mean Newton's method (7) converges to α and satisfies the following error equation:*

$$e_{n+1} = \left(1 - \frac{2^{\frac{1}{p}}\left(1 + (\frac{m-1}{m})^{(m-1)p}\right)^{-\frac{1}{p}}}{m}\right) e_n + O(e_n^2), \tag{9}$$

where $e_n = x_n - \alpha$.

PROOF. Let α be a zero of multiplicity m (i.e. $f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$).
By Taylor expansion of $f(x_n)$ about α we get

$$\begin{aligned}
f(x_n) &= f(\alpha) + f'(\alpha)e_n + \tfrac{1}{2!}f''(\alpha)e_n^2 + \ldots \\
&\quad + \tfrac{1}{m!}f^{(m)}(\alpha)e_n^m + \tfrac{1}{(m+1)!}f^{(m+1)}(\alpha)e_n^{m+1} + \tfrac{1}{(m+2)!}f^{(m+2)}(\alpha)e_n^{m+2} \\
&\quad + O(e_n^{m+3}) \\
&= \tfrac{1}{m!}f^{(m)}(\alpha)e_n^m + \tfrac{1}{(m+1)!}f^{(m+1)}(\alpha)e_n^{m+1} + \tfrac{1}{(m+2)!}f^{(m+2)}(\alpha)e_n^{m+2} \\
&\quad + O(e_n^{m+3}) \\
&= \tfrac{f^{(m)}(\alpha)}{m!}[e_n^m + \tfrac{f^{(m+1)}(\alpha)}{(m+1)f^m(\alpha)}e_n^{m+1} + \tfrac{f^{(m+2)}(\alpha)}{(m+2)(m+1)f^m(\alpha)}e_n^{m+2} + O(e_n^{m+3})], \\
&= \tfrac{f^{(m)}(\alpha)}{m!}[e_n^m + d_{m+1}e_n^{m+1} + d_{m+2}e_n^{m+2} + O(e_n^{m+3})],
\end{aligned} \tag{10}$$

where $e_n = x_n - \alpha$ and $d_{m+i} = \frac{f^{(m+i)}(\alpha)/(m+i)!}{f^{(m)}(\alpha)/m!}$.

Similarly, we obtain

$$\begin{aligned} f'(x_n) &= f'(\alpha) + f''(\alpha)e_n + \tfrac{1}{2!}f'''(\alpha)e_n^2 + ... \\ &\quad + \tfrac{1}{m!}f^{(k+1)}(\alpha)e_n^m + \tfrac{1}{(m+1)!}f^{(m+2)}(\alpha)e_n^{m+1} + \tfrac{1}{(m+2)!}f^{(m+3)}(\alpha)e_n^{m+2} \\ &\quad + O(e_n^{m+3}) \\ &= \tfrac{1}{(m-1)!}f^{(m)}(\alpha)e_n^{m-1} + \tfrac{1}{m!}f^{(m+1)}(\alpha)e_n^m + \tfrac{1}{(m+1)!}f^{(m+2)}(\alpha)e_n^{m+1} \\ &\quad + O(e_n^{m+2}) \\ &= \tfrac{f^{(m)}(\alpha)}{m!}[me_n^{m-1} + (m+1)d_{m+1}e_n^m + (m+2)d_{m+2}e_n^{m+1} + O(e_n^{m+2})] \\ &= \tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}[1 + \tfrac{m+1}{m}d_{m+1}e_n + \tfrac{m+2}{m}d_{m+2}e_n^2 + O(e_n^3)] \end{aligned} \tag{11}$$

Dividing (10) by (11), we get

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= \tfrac{1}{m}e_n\big(1 + d_{m+1}e_n + d_{m+2}e_n^2 + O(e_n^3)\big) \\ &\quad \cdot\Big(1 + \tfrac{m+1}{k}d_{m+1}e_n + \tfrac{m+2}{m}d_{m+2}e_n^2 + O(e_n^3)\Big)^{-1} \\ &= \tfrac{e_n}{m}\big(1 + d_{m+1}e_n + d_{m+2}e_n^2 + O(e_n^3)\big)\Big[1 - (\tfrac{m+1}{m}d_{m+1}e_n \\ &\quad + \tfrac{m+2}{m}d_{m+2}e_n^2 + O(e_n^3)) + (\tfrac{m+1}{m}d_{m+1}e_n + O(e_n^2))^2\Big] \\ &= \tfrac{e_n}{m}\big(1 + d_{m+1}e_n + d_{m+2}e_n^2 + O(e_n^3)\big)\Big[1 - \tfrac{m+1}{m}d_{m+1}e_n + O(e_n^2)\Big] \\ &= \tfrac{e_n}{m}\Big[1 - \tfrac{1}{m}d_{m+1}e_n + O(e_n^2)\Big] \end{aligned}$$

and

$$\begin{aligned} v_{n+1} - \alpha &= e_n - \tfrac{f(x_n)}{f'(x_n)} \\ &= \tfrac{m-1}{m}e_n + \tfrac{1}{m^2}d_{m+1}e_n^2 + O(e_n^3). \end{aligned} \tag{12}$$

By (12) and expanding $f'(v_{n+1})$ about α we obtain

$$\begin{aligned} f'(v_{n+1}) &= f'(\alpha) + f''(\alpha)(v_{n+1} - \alpha) + \tfrac{1}{2!}f'''(\alpha)(v_{n+1} - \alpha)^2 \\ &\quad + \tfrac{1}{3!}f^{(4)}(\alpha)(v_{n+1} - \alpha)^3 + ... \\ &= \tfrac{1}{(m-1)!}f^{(m)}(\alpha)(v_{n+1} - \alpha)^{m-1} \\ &\quad [1 + \tfrac{m+1}{m}d_{m+1}(v_{n+1} - \alpha) + \tfrac{m+2}{m}d_{k+2}(v_{n+1} - \alpha)^2 + ...] \\ &= \tfrac{f^{(m)}(\alpha)}{(m-1)!}(\tfrac{m-1}{m})^{m-1}e_n^{m-1}[1 + O(e_n)] \end{aligned} \tag{13}$$

From (11) we get

$$\begin{aligned}\big(f'(x_n)\big)^p &= \big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\big)^p\big[1+p\Big(\tfrac{m+1}{m}d_{m+1}e_n+\tfrac{m+2}{m}d_{m+2}e_n^2+O(e_n^3)\Big)\\ &\quad+\tfrac{p(p-1)}{2}\Big(\tfrac{m+1}{m}d_{m+1}e_n+\tfrac{m+2}{m}d_{m+2}e_n^2+O(e_n^3)\Big)^2+O(e_n^3)\big]\\ &= \big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\big)^p\big[1+O(e_n)\big],\end{aligned} \tag{14}$$

and similarly from (13) we get

$$\big(f'(v_{n+1})\big)^p = \big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}(\tfrac{m-1}{m})^{m-1}e_n^{m-1}\big)^p\big[1+O(e_n)\big]. \tag{15}$$

It is easy to show that $f'(x_n)f'(v_{n+1})>0$ holds for $n=0,1,2....$ Without loss of generality we suppose that $f'(x_n)>0$ and $f'(v_{n+1})>0$. Replacing (14) and (15), we obtain

$$\begin{aligned}M_p(|f'(x_n)|,|f'(v_{n+1})|)^{-1} &= \big(\tfrac{(f'(x_n))^p+(f'(v_{n+1}))^p}{2}\big)^{-\frac{1}{p}}\\ &= \big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\big)^{-1}\big[\tfrac{1}{2}((\tfrac{m-1}{m})^{(m-1)p}+1)+O(e_n)\big]^{-\frac{1}{p}}\\ &= \big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\big)^{-1}\mu^{-\frac{1}{p}}\big[1+O(e_n)\big]^{-\frac{1}{p}}\\ &= \big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\big)^{-1}\mu^{-\frac{1}{p}}\big[1+O(e_n)\big],\end{aligned} \tag{16}$$

where

$$\mu=\frac{1}{2}\left((\frac{m-1}{m})^{(m-1)p}+1\right).$$

Hence, from equations (10) and (16), we have that

$$\begin{aligned}\frac{f(x_n)}{sign(f'(x_0))M_p(|f'(x_n)|,|f'(v_{n+1})|)} &= \tfrac{f^{(m)}(\alpha)}{m!}e_n^m[1+O(e_n)]\\ &\quad\cdot\big(\tfrac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\big)^{-1}\mu^{-\frac{1}{p}}\cdot\big[1+O(e_n)\big]\\ &= \tfrac{1}{m\mu^{\frac{1}{p}}}e_n\big[1+O(e_n)\big].\end{aligned}$$

Repleacing this in (7) we obtain

$$x_{n+1}=x_n-\tfrac{1}{m\mu^{\frac{1}{p}}}e_n\big[1+O(e_n)\big]$$

or

$$e_{n+1}=\left(1-\tfrac{1}{m\mu^{\frac{1}{p}}}\right)e_n+O(e_n^2),$$

which shows the linear convergence of the root-power mean Newton's method for multiple roots. □

In following we consider the error equation (9). The optimal value of parameter p is such a value for which the asymptotic error constant in equation

(9), $1-\frac{2^{\frac{1}{p}}\left(1+(\frac{m-1}{m})^{(m-1)p}\right)^{-\frac{1}{p}}}{m}$ is minimal. Function

$$\phi(p)=1-\frac{2^{\frac{1}{p}}\left(1+(\frac{m-1}{m})^{(m-1)p}\right)^{-\frac{1}{p}}}{m}$$

is positive and increasing, therefore the minimal value is achieved when p tends to $-\infty$,

$$\lim_{p\to-\infty}\phi(p)=1-\frac{(\frac{m-1}{m})^{1-m}}{m}.$$

Root-power mean, $M_p(a,b)$ brings to $Min(a,b)$ in case when $p\to-\infty$. Therefore, instead of operator M_p in RPM method (7) we can use operator Min. It is easy to see that $Min(|f'(x_n)|,|f'(v_{n+1})|)=|f'(v_{n+1})|$ holds in a case when we approximate multiple zeros. By this way, we coming to the following method

$$\begin{aligned} x_{n+1} &= x_n-\frac{f(x_n)}{f'(v_{n+1})}, \quad \text{where} \\ v_{n+1} &= x_n-\frac{f(x_n)}{f'(x_n)}, \qquad n=0,1,\ldots \end{aligned} \tag{17}$$

which we call *minimum Newton's method* (MM). From the class of RPM methods this method is the most suitable for approximation multiple zeros. From the theorem 2 we can see that all of RPM methods are at least linearly convergent in case of multiple zeros. In table 1 we compare asymptotic error constant of above mentioned RPM methods with Newton's method.

Table 1.

Mult.	AEC				
	NM	AM	HM	GM	MM
2	0.50	0.33	0.25	0.29	**0.00**
3	0.67	0.54	0.46	0.50	**0.25**
4	0.75	0.65	0.58	0.61	**0.41**
5	0.80	0.72	0.66	0.69	**0.51**
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$1-\frac{1}{m}$	$1-\frac{2}{(1+(\frac{m-1}{m})^{m-1})m}$	$1-\frac{1+(\frac{m-1}{m})^{1-m}}{2m}$	$1-\frac{(\frac{m-1}{m})^{\frac{1-m}{2}}}{m}$	$1-\frac{(\frac{m-1}{m})^{1-m}}{m}$

From table 1 we can see that AEC for MM method, in case of double zeros, is equal to 0. That means the convergence in such a case is quadratically. In every cases MM method has the minimal AEC.

Numerical Example. Let $f(x)=x^2\sin(4x)$. It is obviously that $\alpha=0$ is a zero of multiplicity three. We solve the equation $f(x)=0$ by RPM method using different values of the parameter p. The initial approximation, $x_0=0.3$ and the stopping criterion is $|x_{n+1}-\alpha|+|f(x_{n+1})|<10^{-12}$.

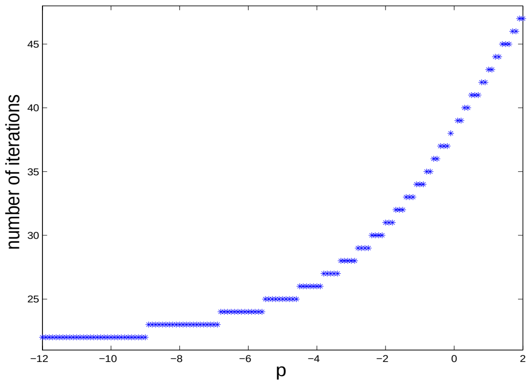

Figure 1.

On figure 1 we show the dependence of iteration number on different chose of parameter p. Decreasing of p implicates decreasing of the iteration number, which is in accordance with above presented theoretical reason, obtained by analysis of the error equation in Theorem 2.

4 Conclusion

The main characteristics of the methods based on aggregation operators, presented in this paper are following: the least third order of convergence for the simple zeros and least linear convergence for the multiple zeros; there exist such value of $p = p_{opt}$, for which value the order of convergence of the RPM method is at least 4; using three functional evaluations in each iteration step; it does not require the computation of the second or the higher order derivatives.

Also, from our analysis, we can conclude that in case when we approximate a multiple zero, it is most suitable to use the MM method, because it has the minimal asymptotic error constant and it is a simple algorithm.

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