Uninorms and Absorbing Norms with Applications to Image Processing

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Abstract: Recently it has been shown that in Image Processing, the usual sum and product of the reals are not the only operations that can be used. Several other operations provided by fuzzy logic perform well in this application. We continue this line of research and we propose the use of a pair consisting of a uninorm and an absorbing norm determined by a continuous, strictly increasing generator instead of the classical sum and multiplication. We define in the present paper pseudo-linear Haar wavelets, and we perform multi-channel decomposition of images. The results show us that pseudo-linear Haar wavelets can be used as an alternative of classical Haar wavelets.

Keywords: uninorm, absorbing norm, Discrete Cosine Transform, wavelets

1 Introduction

In classical Functional Analysis and classical Approximation Theory, the underlying algebraic structure is the linear space structure. The mathematical analysis using nonlinear mathematical structures is called idempotent analysis (see [13]) or pseudo-analysis (see [15], [16]) and it is shown to be a powerful tool in several applications.

Recently we have proposed the same problem in Approximation Theory i.e., is the linear structure the only one that can be used in the classical Approximation Theory? Moreover, are the addition and multiplication of the reals the only operations that can be used for defining approximation operators? All the approximation operators need to be linear? The answer to these questions is negative, and in this sense in [3] max-product Shepard approximation operators are studied. Also, in [5] Pseudo-Linear Approximation Operators are studied from the theoretical point of view and it is shown that even a parametric family of operations can be used for approximation purposes instead of the sum and multiplication of the reals. The idea of the possible usefulness of these operators is coming from Fuzzy Logic. For example in [7], normal forms are regarded as approximations, and this lead us to the idea that sum and addition are not the only operations which can be used in approximation theory.

We will continue in the present paper this line of research and we will focus on a pair consisting of a uninorm and an absorbing norm. Uninorms were introduced by Yager and Rybalov [17] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval. Absorbing norms are a generalization of the well-known median studied in [11]. For a more general treatment of this operator, we refer to [9]. For the absorbing norm (see [2]), a given element has absorbing property, i.e., its composition with any other element gives the absorbing element itself.

We use in the present paper a pair of operations which are distributive one with respect to the other. As a consequence we will use a uninorm based on an additive generator and an absorbing norm based on the same (but multiplicative) generator. For such a pair of operations the distributivity property holds.

One of the classical Image Compression Techniques is using the DCT and the Inverse DCT. By changing the classical operations into a uninorm and absorbing norm as described above, we can define an efficient approximation method, similar to DCT. Then, this approximation method can be used efficiently in order to provide a generalization of the classical DCT-based old jpeg image file compression method.

Wavelets are in the actual state of art in Image Processing, objects used in several applications. Wavelets which are used in image processing employ the linear algebra structure over the reals. Recently, non-linear wavelets based on max-plus algebra structure were successfully applied in image processing (see [12]). These ideas lead us to consider pseudo-linear wavelets in the present paper. Surely, we begin our investigations with a generalization of the most simple wavelet, i.e., we define the pseudo-linear Haar wavelets. We give in the present paper a multi-channel representation of the pseudo-linear Haar wavelets.

After a section in which we describe the algebraic structure given by a uninorm and an absorbing norm, we recall in Section 3 the approximation method based on DCT with a uninorm and an absorbing norm as operations. In Section 4, we present pseudo-linear Haar type wavelets and a multi-channel image decomposition scheme based on them. In Section 5 we present some numerical results which show the effectiveness of the proposed methods. At the end of the paper some conclusions and topics for further research are given.

2 Uninorms and Absorbing Norms

Firstly let us recall the definitions and some properties of uninorms and absorbing norms.

Definition 1 [17] A uninorm $\oplus$ is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e., for all $x \in [0,1]$ we have $x \oplus e = x$.

Definition 2 [2] An absorbing norm $\odot$ is a commutative, associative and increasing binary operator with an absorbing element $a \in [0,1]$, i.e. $(\forall x \in [0,1])(x \odot a) = a)$.

In the present paper we will use the so-called representable uninorms. These are defined as follows. Given $e \in (0,1)$ and a strictly increasing continuous $[0,1] \to \overline{\mathbf{R}}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$, the binary operator $\oplus$ given by

$$x \oplus y = h^{-1}(h(x) + h(y))$$

for any $(x,y) \in [0,1]^2 - \{(0,1),(1,0)\}$, and either $0 \oplus 1 = 1 \oplus 0 = 0$ or $0 \oplus 1 = 1 \oplus 0 = 1$, is a uninorm with neutral element e. The class of uninorms that can be constructed in this way has been characterized in [10].

Let us consider now the operation

$$x \odot y = h^{-1}(h(x)h(y)).$$

The next Theorem shows us that the algebraic structure obtained based on the above described pair of operations.

Theorem 1 Given e and h as above, then the operations $\oplus$ and $\odot$ have the following properties:

(i) The operation $\odot$ is an absorbing norm, having e as absorbing element.

(ii) The element $e' = h^{-1}(1)$ is neutral element w.r.t. $\odot$.

(iii) $\odot$ is distributive with respect to $\oplus$.

(iv) Any $x \in (0,1)$ has an opposite w.r.t. $\oplus$.

(v) Any $x \in (0,1) - \{e\}$ is invertible w.r.t. $\odot$

Proof (i) It is easy to check that $\odot$ is commutative, associative and increasing. Also, by direct computation it follows that

$$x \odot e = h^{-1}(0) = e, \forall x \in (0,1),$$

that is, e is absorbing element.

(ii) It is easy to check that

$$x \odot e' = h^{-1}(h(x) \cdot 1) = x.$$

(iii) The distributivity law is obvious.

(iv) We take $\ominus x = h^{-1}(-h(x)) \in (0,1)$. Then by direct computation follows $x \oplus (\ominus x) = e$.

(v) Let $x^{-1} = h^{-1}\left(\frac{1}{h(x)}\right) \in (0,1) - \{e\}$ for any $x \in (0,1) - \{e\}$. Then it is easy to check that $x \odot x^{-1} = e'$.

Notation We denote $x \ominus y = h^{-1}(h(x) - h(y))$. It is easy to observe that we have $x \ominus y \oplus y = x,\ \forall x, y \in [0,1]$.

Remark Nullnorms [6] are absorbing norms fulfilling some boundary conditions. In our case the boundary conditions are not satisfied. Instead, on the boundaries, if we pass to limit, we get

$$x \odot 0 = \begin{cases} 1 \text{ if } x < e \\ e \text{ if } x = e \\ 0 \text{ if } x > e \end{cases}$$

and

$$x \odot 1 = \begin{cases} 0 \text{ if } x < e \\ e \text{ if } x = e \\ 1 \text{ if } x > e \end{cases}$$

Let us also observe here the strong relationship between pseudo-analysis and our proposed pairs of operations. The algebraic structure induced by the above described pair of operations is a semiring as in the case of pseudo-analysis. Also, the properties (iv) and (v) show us that the algebraic structure induced by the above pair of operations on $[0,1]$ interval is very close to the structure of a commutative field. This remark allows us to use these pairs of operations instead of the standard addition and multiplication of the reals.

3 DCT based on a Uninorm and an Absorbing Norm

The pseudo-linear DCT was proposed in the recent paper [4]. Let us consider a continuous target function $f : [a,b] \to [0,1]$. Let also, $f_i \in [a,b]$ be data associated to f. The idea of defining nonlinear approximation operators is very simple. We change the addition into a uninorm and multiplication into the corresponding absorbing norm, in the approximation operators, i.e. the general discrete form of such an operator is

$$P_n(f,x) = \bigoplus_{i=0}^{n} B_{n,i}(x) \odot f_i,$$

where $B_{n,i}(x) : X^2 \to [0,1]$ are some given continuous functions, $i = 1,\dots,n$.

Let us denote by $C([a,b])$ the space of continuous functions $f : [a,b] \to \mathbf{R}$.

The operator $P_n : C([a,b]) \to C([a,b])$, $P_n(f,x) = \bigoplus_{i=0}^{n} B_{n,i}(x) \odot f_i$, is continuous and satisfies the property

$$P_n(\alpha \odot f \oplus \beta \odot g, x) = \alpha \odot P_n(f,x) \oplus \beta \odot P_n(g,x).$$

In the present paper we propose a generalization of the DCT method. So, let us consider the functions

$$A_{n,i}(x) = \cos\frac{\pi(2x-1)(i-1)}{2n}, \quad i = 1,\dots,n.$$

The inverse DCT is

$$T_n(f,x) = \sum_{i=1}^{n} A_{n,i}(x) f_i,$$

where f_i denote the classical DCT of the target function f. Let us transform the functions $A_{n,i}$ by h^{-1}, i.e., we let

$$B_{n,i} = h^{-1} \circ A_{n,i}$$

provided the the range of $A_{n,i}$ is contained in the domain of h^{-1}. Let $P_n(f,x)$ given as in (oper). Then $P_n(f,x)$ will be referred as pseudo-linear inverse DCT.

With respect to the proposed approximation operator, in the next Theorem we deduce an error estimate.

Theorem 2 Let $P_n\left(f,x\right)$ be the operator given above and the operations $\oplus$ and ? generated by a generator $h,$ where h is bi-Lipschitz on any closed subinterval $[a,b] \subset [0,1],\ \ 0 < a < b < 1$. Then the following Jackson-type estimate holds true

$$| P_n\left(f,x\right) - f(x) | \leq C\omega\left(f,\frac{1}{n}\right), \forall x \in [a,b]$$

C being some absolute constant.

It is now straightforward the following corollary which is a Weierstrass-type theorem for the proposed pseudo-linear inverse DCT.

Corollary 1 Any continuous function $f\ :[0,1] \to [0,1]$ can be arbitrarily closely approximated by trigonometric approximation operators based on a uninorm and an absorbing norm given by a generator as above.

However all the properties are straightforward and however the method is based on the classical approximation result, it worth studying, because firstly, the result obtained for approximation of the target function is different and also they are different seen as approximation operators, one being linear and the other is not linear.

4 Haar Wavelets based on a Uninorm and an Absorbing Norm

4.1 One-dimensional Case

We define in the present section a Haar-type wavelet based on a pair consisting of a uninorm and an absorbing norm.

Given an input signal $x\ :\ \mathbf{Z} \to \mathbf{R},$ we split it into $x_0(n)$ and $x_1(n)$ the odd and even samples. The classical two-channel Haar wavelet transform is given by the Analysis and Synthesis operations based on a perfect reconstruction condition.

4.1.1 Analysis

The analysis operation gives the highpass and lowpass subband samples as:

$$\begin{pmatrix} x_0(k) \\ x_1(k) \end{pmatrix} \to \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix},$$

where

$$\begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{pmatrix} \frac{x_0(k)+x_1(k)}{\sqrt{2}} \\ \frac{x_0(k)-x_1(k)}{\sqrt{2}} \end{pmatrix}$$

4.1.2 Synthesis

We apply the same operation as in analysis step to the signal y, and it is easy to check that we reobtain the original signal x.

Let us now propose the pseudo-linear version of the above described Haar wavelet.

4.1.3 Analysis

The analysis operation gives the high pass and low pass subband samples as a pseudo-linear transformation:

$$\begin{pmatrix} x_0(k) \\ x_1(k) \end{pmatrix} \to \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix},$$

where

$$\begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} = \begin{pmatrix} (x_0(k) \oplus x_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ (x_0(k) \ominus x_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$$

4.1.4 Synthesis

We apply the same operation as in analysis step to the signal y, and it is easy to check that we reobtain the original signal x. Indeed, it is easy to check that

$$\begin{aligned} &(y_0(k) \oplus y_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= ((x_0(k) \oplus x_1(k)) \oplus (x_0(k) \ominus x_1(k))) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= (x_0(k) \oplus x_0(k)) \odot g^{-1}\left(\frac{1}{2}\right) = x_0(k). \end{aligned}$$

and that

$$\begin{aligned}
&(y_0(k) \ominus y_1(k)) \odot g^{-1}\left(\frac{1}{\sqrt{2}}\right) \\
&= ((x_0(k) \oplus x_1(k)) \odot (x_0(k) \odot x_1(k))) \odot g^{-1}\left(\frac{1}{2}\right) \\
&= (x_1(k) \oplus x_1(k)) \odot g^{-1}\left(\frac{1}{2}\right) = x_1(k).
\end{aligned}$$

4.2 Bidimensional Case

As the classical bidimensional linear Haar transform is obtained by applying two times the one dimensional transform on horizontal and vertical directions, the pseudo-linear case will be similar. We define in what follows the 4-channel pseudo-linear Haar wavelets in two dimensions.

4.2.1 Analysis

The analysis operation is given as follows:

$$\begin{matrix} x(2n) & x(2n^+) \rightarrow & \psi(n) & \omega_h(n) \\ x(2n_+) & x(2n_+^+) \leftarrow & \omega_v(n) & \omega_d(n) \end{matrix},$$

where

$$\begin{aligned}
\psi(n) &= (x(2n) \oplus x(2n^+) \oplus x(2n_+) \oplus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right) \\
\omega_h(n) &= (x(2n) \oplus x(2n^+) \ominus x(2n_+) \ominus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right) \\
\omega_v(n) &= (x(2n) \ominus x(2n^+) \oplus x(2n_+) \ominus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right) \\
\omega_d(n) &= (x(2n) \ominus x(2n^+) \ominus x(2n_+) \oplus x(2n_+^+)) \odot g^{-1}\left(\frac{1}{2}\right).
\end{aligned} \tag{1}$$

4.2.2 Synthesis

The synthesis operation is as in the classical case the same as the analysis operation, but applies to the result of the analysis operation. It is easy to check that the perfect reconstruction requirement is satisfied.

5 Image Processing Experiments

5.1 Experimental Results for Pseudo-Linear DCT

We perform in this section some preliminary image processing experiments by using DCT-based approximation method described above, based on uninorms and absorbing norms. Let us regard the following results as a competition between addition, multiplication pair and other pairs of operations in the framework of Image Processing.

The compression method is the classical direct DCT, i.e., in the compression step we calculate the DCT coefficients of the original image and we delete most of them according to a given mask. The block dimension for DCT in the proposed experiments was 8×8 pixels and for the same image we have used always the same mask, in order to compare the results under the same experimental conditions.

The decompression step is as follows. We compute the pseudo-linear inverse DCT by using the coefficients obtained in the compression step, for different parametric operations.

As generators for the uninorm and absorbing norm we have used

$$h(x) = \ln \frac{x^a}{1 - x^\alpha}, \tag{2}$$

which generates a parametric family of operations. The neutral element of the uninorm in this case is $\frac{1}{2^{\frac{1}{\alpha}}}$. This parametric family, for $\alpha = 1$ contains the famous 3 PI operation.

We present two experiments.

In the first experiment, original image Lenna is compressed by using the conventional method and the proposed method, the compression rate being $\frac{6}{64}$. In Fig. 2 the decompression results are presented. However the visual quality of the images is similar, if we compare the RMSE in the proposed two methods, we observe that the conventional method gives $RMSE = 12.2254$, while the proposed method gives $RMSE = 12.0041$. The dependence of the RMSE with respect to the parameter α of the generator in eq. (gener) is shown in Fig. 3. The best value of the parameter α in this experiment was $\alpha = 0.85$.

Figure 1
Original Images Lenna and Airplane

Figure 2
Decompression results: classical DCT(left), proposed method(right)

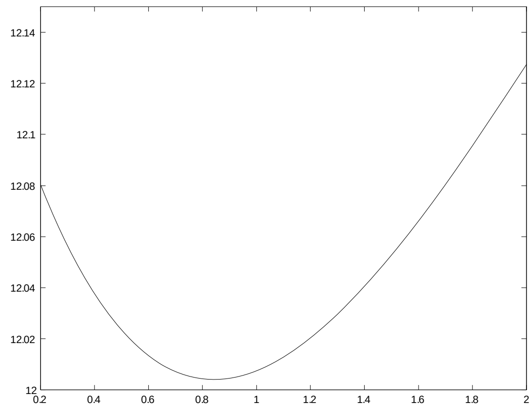

Figure 3
Dependence of the RMSE w.r.t. the parameter α for image Lenna

In the second experiment, Airplane image is compressed with the compression rate $\frac{10}{64}$. In Fig. 4 the decompression results of the conventional and proposed method can be compared. The RMSE in the case of using the classical method is 12.2281, while with the proposed method $RMSE = 11.9009$. The dependence of the RMSE with respect to the parameter α of the generator is shown in Fig. 5. The best value of the parameter α in this experiment was $\alpha = 1.35$.

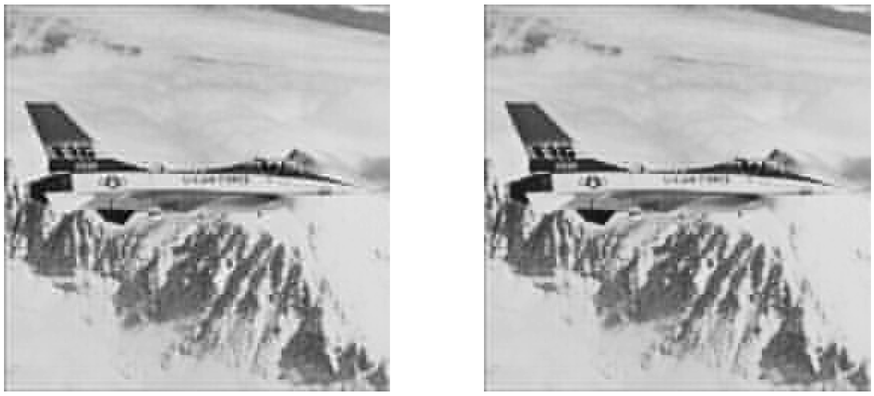

Figure 4

Decompression results: classical DCT(left), proposed method(right)

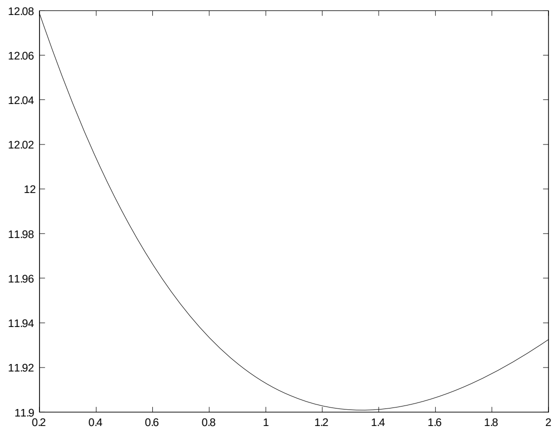

Figure 5

Dependence of the RMSE w.r.t. the parameter α for image Airplane

5.2 Multi-Channel Image Decomposition

We perform in this section multichannel image decomposition by using Haar wavelets based on uninorms and absorbing norms, defined as above. These experiments show us the possible usefullness of the proposed method as an alternative to Haar wavelets. Surely, JPEG 2000 compression standard uses more sophisticated wavelets and their study is subject of future research.

In the following experiments image Lenna is decomposed until the second decomposition level by using the same generator (2) for different values of the parameter a (see Figs. 7, 8). The decomposition result is represented as described in Fig. 6, where the upper left corner corresponds to the component ψ in (1), while the lower left, lower right and upper right correspond to $\omega_v, \omega_d, \omega_h$ respectively of the pseudo-linear Haar wavelet. Since the exact reconstruction of the original Lenna image is possible by using the given image decomposition results and the pseudo-linear Haar wavelet again, the effectiveness of the proposed method is shown.

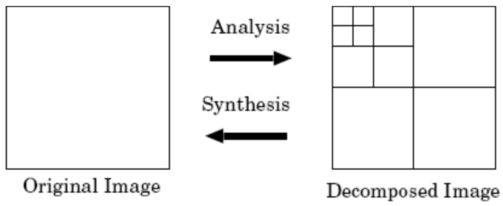

Figure 6

Analysis and Synthesis operations used for image decomposition

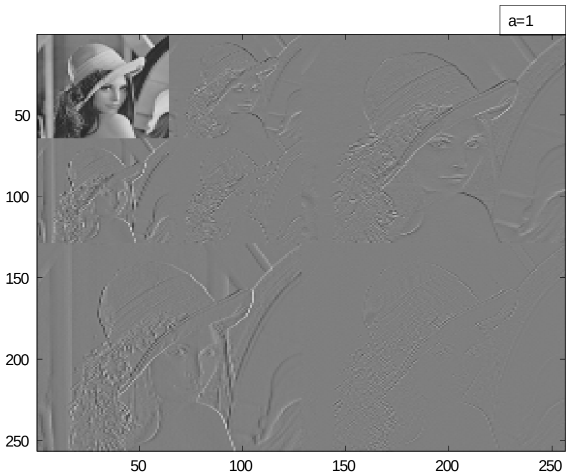

Figure 7

Multichanel decomposition of the image Lenna based on the proposed method, parameter of the uninorm-absorbing norm a=2

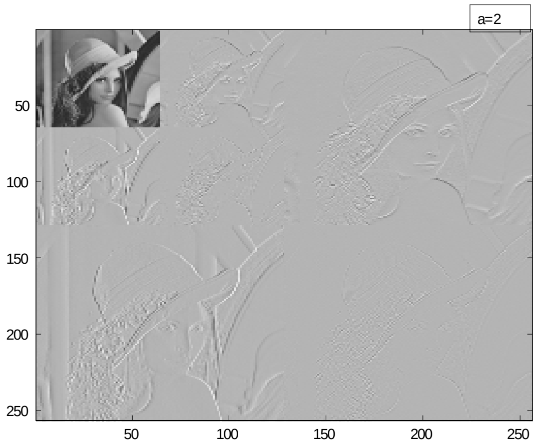

Figure 8
Multichanel decomposition of the image Lenna based on the proposed method, parameter of the uninorm-absorbing norm a=2

It is interesting to observe that the basic colour in the lower components increases together with the value of the parameter. This is due to the fact that the colour in the lower component should be near to the null element of the operation which is now not 0 as in the classical case.

The results show that the pair consisting of a uninorm and absorbing norm can be used in Image processing as alternatives of sum and product operations.

If we regard the results presented above as a competition between the classical and other operations we can observe that the addition and product are outperformed in this experiment by pairs consisting of a uninorm and an absorbing norm.

Conclusions

We have proposed the use of a pair of generated uninorm and absorbing norm in Approximation Theory and in Image Processing. The theoretical study shows that the main properties are conserved.

As a conclusion of the experiments proposed in the previous section it is easy to see that the sum-product based approximation is sometimes outperformed by our proposed method.

As topics for future research let us mention the use of the proposed approximation operators for noise reduction or debluring of images. Surely more efficient image compression methods can be imagined. As a next step we would like to study Daubechies-type pseudo-linear wavelets based on uninorms and absorbing norms.

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